Intuition and rigor in mathematics instruction

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1. Intuition in the didactical tradition

There is probably no adequate translation for the German concept of *Anschauung*. It has connotations that can only be understood in light of its history in philosophical and didactical discourse.

1.1 In the 19th century, the concept of *Anschauung* (intuition) was marked by a fundamental ambiguity. On the one hand, *Anschauung* meant to look concretely at something, in the sense of empirical perception; on the other hand, people always had a type of *abstract intuition* in mind that signified a purely mental representation of ideal objects or structures. This double meaning of *Anschauung* can already be found in Pestalozzi (1746-1827): While emphasizing the pedagogical significance of sense perception, he nevertheless formulated the pedagogical demand "to liberate *Anschauung* itself from the limits of mere sensation" (quoted in Timerding, 1912, p. 11; our translation). With this view, Pestalozzi followed Kant, who had systematically distinguished between empirical and pure (a priori, nonempirical) intuition, defining mathematics as that science which constructs its concepts in the pure intuition of space and time (Kant, 1781, pp. 741-742). Accordingly, Pestalozzi considered the concept of number to be a creation of the mind that could not be derived from experience. The role of experience was only to provide an occasion to actualize numbers (see Timerding, 1912, pp. 13ff.).

Whatever views the German-speaking pedagogical authors of the 19th century may have held, it was indisputable among them that, though mathematics instruction should start from experience, its main objective should be to further a nonempirical intuition beyond mere perception. Frequently the difference between the two kinds of *Anschauung* was characterized as the contrast between *äußere* and *innere Anschauung* (external and internal intuition). Tillich (1780-1807), among others, considered *innere Anschauung* to be a mental activity (Tillich, 1805; see Jahnke, 1990, p. 94).

According to Herbart (1776-1841), the aim of mathematical education was to transform the *rohe Anschauung* (raw intuition) of adolescents into a *reife Anschauung* (mature intuition) (Herbart, 1802; Jahnke, 1990). Subsequently, Herbart developed this idea into his conception of the *five formal steps* of teaching - preparation, presentation, association, generalization, and application - which became very influential in elementary school pedagogy in the second half of the 19th century, but only in narrow and naive interpretations.

In a similar sense, Diesterweg (1790-1866) emphasized that the basic objects of geometry are given only in our internal intuition and "that they have to be brought to clear consciousness by means of external experience" (Diesterweg, 1828, p. 260; see Schmidt, 1985, pp. 10ff.; Winter, 1984b).

1.2 This pedagogical emphasis on internal (mental) intuition prevalent in the 19th century can only be understood if it is seen in the context of contemporary science and philosophy. The hermeneutic view of the sciences promoted the idea that intuition is not a passive mode of cognition but always involves an active effort of interpretation. Just as interpreting a text requires *imagination*, inferring an intended mathematical object from a visual representation presupposes an inner *productive power*. Therefore, in the didactics of descriptive geometry, stimulating the internal intuition was a primary goal (Gugler, 1860), since deriving complex, three-dimensional objects from two-dimensional representations requires an active, productive effort.

On the other hand, projective and algebraic geometry provided a mathematical background for reflecting on a type of intuition beyond spatial representability. The overwhelming powers of mathematical synthesis arising from the introduction of objects that cannot be interpreted empirically (infinite points, lines, and planes; imaginary elements) gave rise to the ideal that *truly geometrical reasoning* should liberate itself from the awkwardness of individual figures and advance without reference to concrete drawings (see Chasles, 1839, pp. 205ff.).

1.3 With the emerge of scientism (under a neo-Kantian influence) in the second half of the 19th century, the aims of education at the *Gymnasium* were determined more and more by content, and the concept of *Anschauung* became pedagogically marginal. Now the subject was *Anschauung und Strenge* (intuition and rigor), and it served to restrict intuitive thinking to propaedeutic exercises. Even Felix Klein's reforms, which led to a revival of the debate on *Anschauung*, were, in principle, not successful in overcoming this restriction.

Only in the elementary school (*Volksschule*) did the concept of *Anschauung* remain an essential reference point for pedagogical work. In this area, a tendency emerged and consolidated that has influenced elementary school teaching up to this day: The visual and physical interpretations of the elementary arithmetical operations served as a basis for decomposing the number concept into numerous different (and frequently only empirical) aspects, which afterwards were to be taught through step-by-step procedures. Already by the end of the 19th century, warnings were given against the negative consequences of this exaggerated "methodologization."

1.4 The different pedagogical conceptions of elementary school and *Gymnasium* remained decisive until the 1960s. In particular, important and fruitful ideas were developed in the primary school that were devoted to an intuitive way of teaching mathematics (see Griesel, 1985). The most recent turn in the discussion of *Anschauung* in Germany (and probably world-wide) was connected with the reform movement of the "new math" and can be characterized best by referring to Bruner's (1960) conception of *fundamental ideas*. According to this conception, enactive, iconic, and symbolic representations provide different - though in a didactical sense equally legitimate - modalities for depicting mathematical knowledge. Thus, it seemed to be possible to teach the fundamental ideas of mathematics to students of any age in an adequate manner. This view offered the theoretical and practical opportunity of linking again the didactics of the elementary school to that of the *Gymnasium*.

The so-called *method of operators* provides an instructive example of this approach. Originally, this conception emerged in the teaching of fractions, the fractional operators being illustrated as arrows or, more drastically, as machines. Operators may be viewed as functions, and since almost every mathematical idea can be subsumed under this notion, the function concept was subsequently made a guiding principle for all mathematics teaching from the primary school until the end of the *Gymnasium*. For about ten years, arrow diagrams played a prominent role in German textbooks in many diverse contexts.

Because of unfavorable experience (see Gerster, 1984; Padberg, 1986), similar to that of other countries, the original hope was abandoned that such visualizations could promote an outward unification of the mathematics curriculum (as a substitute, so to speak, for the concept of function). But of course, as mathematical tools, arrow diagrams are still being used very efficiently and fruitfully in many fields and on various levels. Engel's (1975) probability abacus is an easily accessible example.

Today, most educators consider the construction of visualizations no longer to be a separate field of work but rather, as a rule, as something to be incorporated into the development of large-scale teaching conceptions. A remarkable exception is provided by the workshops on *visualization in mathematics* held each year in Klagenfurt (Kautschitsch & Metzler, 1982-1989).

In recent years, the increasing availability of computers has presented completely new opportunities for visualization, thus changing the situation once more. At present, it is difficult to anticipate the long-term implications for mathematics instruction.

2. Mathematical rigor in science and in school

2.1 With the development of calculus into the dominant mathematical subject in the senior classes of the *Gymnasium*, the conflict between intuition and rigor was transferred, in a modified manner, from 19th century mathematics (Volkert, 1986) into the 20th century school. While mathematical science has worked out a feasible and adequate way of representing its knowledge in a semi-formal language, in school the conflict still persists. There the problem cannot be solved so long as many teachers and educators understand mathematics - perhaps only subconsciously - as that content-independent form of deduction which they had taken pains to acquire during their studies. They link it, in principle, to the supposedly rigorous algebraic (symbolic) mode of doing mathematics, while viewing geometrical arguments, in an alleged contrast, as only intuitively acceptable.

The discussion about the respective relevance of intuition and rigor in mathematics instruction that has been going on in Germany since the 1950s can be seen from some issues of the journal *Der Mathematikunterricht* with the titles "Geometrical Means in Calculus" (Freund 1960; our translation) and "Intuition and Rigor in Calculus" (I: Baur, 1957; II & III: Kropp, 1968 & 1969; IV: Blum & Kirsch, 1979a; our translation). In Issues I to III (especially in II and III), the alleged conflict is solved by conceiving intuitive teaching as a method that is to be gradually replaced in the course of the calculus curriculum by an increasing rigor, being reduced later to serving only as *intuitive background* (Baur, 1957, p. 4). After this emphasis on rigor in the 1960s and 1970s, beginning with Kirsch (1960) and increasingly in Issue IV, the independent importance of intuitive concepts is elaborated, at least for basic calculus courses. The authors of that issue, however, had to defend themselves against doubts about their mathematical competence (Blum & Kirsch, 1979a, p. 3).

This debate apparently no longer takes place today. Although the emphasis on rigor did not resist the reality of school, and collapsed into itself, one can assume that many teachers of senior classes still feel bound to a certain ideal of rigor that alone gives them the necessary safety in their subject. They consider intuitiveness as a concession to their students' limited abilities.

2.2 Rigor, as a measure of the precision with which the rules, prescriptions, and standards of some field are observed, is basically a socio-cultural category. It may refer to the rules of a game, administrative regulations, the standards for scientific methods, or the forms of discursive thought, reasoning, and discussion. Mathematics instruction, too, has to promote and cultivate these forms of intellectual activity. Mathematical rigor, however, is of a specific character and cannot be transferred to other fields without substantial modifications. This specific character is not so much a consequence of a complete rigor that, in didactical and philosophical statements about mathematics, is frequently assumed to be realizable. Rather it follows from the specific character of the mathematical objects and from the social frame of mathematical practice.

The ideal of mathematical rigor is modelled according to the concept of a completely formalizable theory developed in formal logic, which comprises both a stock of uninterpreted symbols and certain elementary propositions (axioms) from which further propositions are formed that obey predefined rules in a purely mechanical way. Thus, the idea of rigor implies both working in the *formal mode* with uninterpreted concepts and the *completeness* of deduction. However, mathematicians consider these features of reasoning only as reference points and not as absolute norms that must be fulfilled completely.

The formal construction of arithmetic, together with the consequent deduction of seemingly simple theorems such as "1 + 1 = 2," given by Whitehead and Russell in the three volumes of their *Principia Mathematica* (1910-1913), show that in mathematical *practice* completeness is *unattainable*, and *undesirable*. Proofs that in the sense of formal logic have gaps are thus not automatically considered as violating mathematical rigor. Whether a proof is judged mathematically incomplete depends on many conditions that are to some extent subject to historical change: Were the gaps left intentionally? Are they indicated as such? Can they be closed? How much effort is needed to close them? What is the author's reputation? What are the actual paradigms of mathematics?

From the logician's point of view, the "average" professional mathematician does not work with uninterpreted concepts. His or her language (about sets and functions) is not formalized but rather filled with interpreted concepts and phrases. Furthermore, mathematicians, too, need, whether they are aware of the fact or not, content-related, intuitive interpretations for the production and reception of proofs. That is why a standard has emerged in mathematics that might be called *formally mathematical* and that is a weaker variant of *formally logical*.

Whether a proof is accepted as mathematically rigorous cannot be decided definitely but is subject to a large measure of judgement. This uncertainty causes the expert to be dependent on other experts, that is, on the mathematical community. This dependency cannot be removed by the use of machines either, and it is so sweeping that *a proof only becomes a proof* if and because the community accepts it as such. The social element inherent in the processes of establishing proofs was introduced as a decisive argument into the didactical debate by Thom (1970); later it was analyzed in detail for the domain of mathematics by Hanna (1983, pp. 70ff.). According to Wittmann and Müller (1988), among others, it should be taken into consideration in the domain of teaching, too. That would be also pedagogically advantageous, as it would reduce the teacher's central role in several respects.

For the mathematician, the formal point of view is not an end in itself. Rather, it serves, firstly, to ensure objectively in a deductive manner and to communicate *mathematical* knowledge (not absolute truths about the world). Secondly, many mathematical concepts can only be expressed, analyzed, and reflected by means of a formal and symbolic representation. In the history of mathematics, the idea of infinity is a paradigmatic case, but many concepts of elementary mathematics are also affected by this condition. And thirdly, formalization also has the critical function of uncovering inadmissible conclusions and tacit assumptions.

Hence, formal representation and deduction are necessary components of elementary mathematical activities, too, and thereby the problems of learning mathematics are increased enormously. On the one hand, it is very difficult for the learner to establish a cognitive distance from everyday experience, to execute the de-interpretation of concepts, and to assume even a moderately formal attitude. On the other hand, nonformal deductions often require, as all experience has shown, a high degree of training, as they lack the reduction of complexity linked to formal notations. Therefore, it is quite unreasonable to demand a solely intuitive way of doing mathematics in the classroom and to dispense completely with formal representations and deductions.

2.3 In our opinion, an adequate conception by which the interrelations between intuition and rigor can be understood theoretically and shaped practically emerges from realizing mathematical theories, even if they are worked out in a rather formalized way, as necessarily *bound to their domain of intended applications* (in the sense of Sneed, 1971; see also Jahnke, 1978). According to the prevailing opinion, mathematicians claim that the domains of intended applications for their theories are unbounded in principle. Usually, this *global character* is achieved by taking possible restrictions in the range of a theorem as part of its assumptions, thus formally imparting to it its universal nature. This method of forcing the global character is supplemented by the mathematicians' striving for "strong" theorems - that is, for assertions that reach as far as possible together with assumptions that are as weak as possible. Nonexpert readers are usually unable to recognize (and they are often actively supported in their ignorance by the authors' tendency to erase the traces of their work) that mathematical theorems, at least as they come into being, are mostly related very precisely to a certain spectrum of problems and test cases that are to be investigated with their help.

Professional users of mathematics do not need the mathematicians' claim of global validity. And in school, anyway, mathematical concepts are treated only with strongly limited ranges. In this regard, instruction differs fundamentally from science. In school, doing mathematics always means to be bound to concrete, very restricted domains of intended applications. (Here the notion of application is not to be understood in a narrow sense; it also comprises mathematical topics, contexts of use, etc.) By varying such domains (e.g., the transition from two- to three-dimensional space), students can possibly acquire a first awareness of the problem of globality, but as they lack the intensive cognitive socialization of the professional mathematician, they are not able to grasp the substance of this idea. Yet abandoning completely the claim for globality is not simply a necessary evil; it is justified because in school, unlike in science, dealing with mathematics is, in the first place, to promote *subjective insights*.

Beyond these reasons, this fundamental deviation from the paradigm of mathematics is by no means a (necessary and legitimate) shortcoming. The assumption of narrow domains of intended applications is instead a prerequisite for intellectually honest reasoning on the school level, as it also narrows considerably the domain of potential gaps in the argument, in quality as well as in quantity, and enables the students to direct their reasoning along content lines and not just logically. In the tradition of Thom (1970), Freudenthal (1973), and others, we believe that mathematics instruction should deal with meaningful statements about meaningful objects. This belief gives rise to the following description of a concept of rigor that in our opinion is appropriate for school:

Thesis 1: In school, mathematical reasoning should explicitly take place as content-related reasoning. With this qualification, completeness is preserved as a local requirement. Consequently, in school rigor can only mean sound reasoning in restricted domains of intended applications by means of interpreted concepts.

With this specification we deliberately leave wide scope for what can be understood by "adequate rigor." In fact, it is always a practical problem to develop an appropriate orientation for the particular didactical situation. Moreover, the idea of rigor is subject to historical changes in the paradigms of mathematics itself as well as of those of mathematics education, and it depends, of course, on the special topic (e.g., geometry, algebra, calculus) and on the students' stages of development. At any rate, that idea of rigor which in our opinion is adequate (not only) for school allows one *to do rigorous mathematics also in the intuitive mode of reasoning*. Examples at different levels, including explicit discussions of the problem, are to be found in Kirsch (1979), Wittmann & Müller (1988), and Blum & Kirsch (1991), among others.

Situations continually do occur, however, where the students cannot finally judge independently the validity of an argument and the teacher's professional authority is needed. This discrepancy produces a cognitive asymmetry between the teacher's and the students' roles. And it has always been an open problem to reconcile this asymmetry with the pedagogical demand for the principle of autonomous learning.

We want to stress that within such a framework of interpreted mathematics there is place as well for formal representations and formal reasoning: Their function is to sharpen locally one's own thinking, and to serve as a means of reflection. The students, too, should be given an impression of the logical power of de-interpreting mathematical objects and of removing the restrictions from the domains of intended applications, with which their universes of thinking (should) largely coincide (e.g., by operating with "meaningless" letters or by raising apparently absurd counterexamples to propositions that seem to be obviously true).

3. Intuition and learning

3.1 If one tries to explain intuition in a positive way, the following three aspects prove to be essential:

1. *Intuition* (as a mode of thinking),
2. *Interpretation* (activities directed at concepts, and the results of such activities), and
3. *Intention* (objectives, purposes, motives).

These aspects are loosely complementary to those components of rigor that we discussed in Section 2 (completeness, formalization, global character).

Wherever *intuitive thinking* takes place, it serves to bridge logical gaps in a (by narrow standards) *generous* manner. Intuitions can be *anticipatory* in the course of solving a problem autonomously (e.g., finding a proof), as well as *affirmative*, when working on a given solution of a problem (e.g., studying a given proof) (Fischbein, 1987).

In order to unfold, these two aspects of intuition presuppose a familiar "cognitive space," and this space is prepared by the interpretation of the concepts involved. First, we understand *interpretation* as ascribing meaning to concepts, primarily by anchoring them in the real world (in a wide sense). But the phrase "interpretation" also includes the embodiment of concepts in some mathematical situation, where they can be provided with a new meaning or with some meaning, on the grounds of a new context, and where the direct reference to the real world (taken literally) can vanish.

Normally the recourse to the real world takes place on three levels, and each time some intended applications play a constituting role: Firstly, when *concepts are created* (historically) or *developed* (practically) in order to solve some problem (which is the case with many geometrical concepts in an elementary way; see Bender & Schreiber, 1985); secondly, in *didactical situations* in the classroom, which, by definition, have a teleological character; and, thirdly, in those real-world contexts that are designed by curriculum developers as frameworks for teaching mathematical contents.

The two aspects *interpretation* and *intention* are elements of *intuition*, and they constitute it at the same time. It is important to note that they are closely interrelated. If a teacher disregards this interrelation they often produce unfruitful conflicts for the students. For example, the well-known geometrization of the binomial formula by rectangular areas offers an appropriate picture, but it may distract from the intentions actually pursued, namely making the structures of the formula transparent. For these intentions a visualization like the following may be better:

$$
\begin{aligned}
(a+b)^2 &= a^2+b^2 &&+ 2ab\\
(a+b+c)^2 &= a^2+b^2+c^2 &&+ 2(ab+ac+bc)\\
(a+b+c+d)^2 &= a^2+b^2+c^2+d^2 &&+ 2(ab+ac+ad+bc+bd+cd)
\end{aligned}
$$

and so forth.

It offers directly insights that immediately refer to the structure of the expression.

3.2 The obvious discrepancy between the uninterpreted (abstract) concepts of mathematics and the interpreted (concrete) concepts of humans (in particular, novices) makes mathematics teaching a difficult job. Interpreted concepts are embedded in much richer contexts, which are (didactically) more or less productive and which are not merely analogous mappings of mathematical relations.

How to cope with this richness; how to judge the students' conceptions; how to create, develop, or eliminate them (if necessary) - these issues are usually not touched upon in the classical, content-oriented teacher education at the university. But in the discipline of didactics of mathematics, too, considerable deficits exist concerning these questions.

Plenty of examples and techniques of visualizations are available. But they are often bound up with the principle of chopping mathematics lessons into tiny methodological pieces by which teachers and textbook authors seek to overcome the primacy of abstract rigor, in which they still believe either overtly or subliminally (see the textbook critique by Keitel, Otte, & Seeger, 1980, chap. 7). With this over-methodization, mathematics education runs the risk of burying the central ideas of the discipline (in the sense of Schreiber, 1983) and - paradoxically - even of obstructing learning processes.

3.3 With regard to the present importance of visual communication by way of plane media like paper, television screens, computer screens, and so forth, visualizations (and in particular, geometrical, two-dimensional relations) are the prevailing type of interpretations, either actually realized or only prepared for the students to imagine. They are easily available (which has been so all along, but the possibilities have been immensely extended with the help of computer technology). Real space is part of everyone's elementary experience. On the one hand, its structure is rich enough for generating or representing even complex mathematical concepts and relations; on the other hand, it is sufficiently poor to limit extra-mathematical meaning to the desired scope. But above all, students are seemingly able to understand such visualizations without special prerequisites.

Apart from visualizations, there are interpretations in many other fields. The first is elementary physics. Beyond that, there are many contexts in everyday life and in vocational life, not to mention technology, natural sciences, economics, and so forth, that can be utilized for interpretations (Blum & Törner, 1983, p. 92). But the more elaborate these situations are, the more the students, as well as the teacher, have to provide specific knowledge, and they run the risk of being distracted from those (mathematical) concepts they were originally interested in.

It is a widespread fundamental error to assign automatic intelligibility to pure visualizations (and to other embodiments in everyday life) because they often do not require any specific knowledge, even about geometry. But as is shown by much teaching experience and by empirical research (e.g., by the investigations of Schipper, 1982, in the paradigmatic field of primary school arithmetic):

Thesis 2: Interpretations, even geometrical ones, are not "pictures speaking by themselves." They can fulfill their purposes only on the basis of the learners' active construction of meaning. In each case, they, as well as their relations to other interpretations and to the underlying concepts, have to be learned anew.

Devices meant to facilitate mathematics learning (namely interpretations) actually seem to impede it at first (this amazing observation was also made by Kirsch, 1977). But it is just the additional intellectual effort needed for the constitution and transfer of meaning that establishes the basis for the desired success. Dealing with the preliminary understanding and opinions of other participants in the discourse is another essential part of the constitution of meaning, and at the same time, interpretations, again, are needed as *formats* (in the sense of Bruner) for such discussions (Fischer & Malle, 1985; Krummheuer, 1989b).

We now want to illustrate our second thesis with the example of the *arithmetic mean*, when interpreted as the supporting point of a beam with several loads in balance, that is, as the center of gravity. The interpretation of concepts and relations by means of gravity is the basis of a whole class of far-reaching mathematical considerations; for instance: Archimedes' determinations of areas; the instrumental construction of the Torricelli point of a triangle (to be found in Lietzmann, 1959, p. 98); Winter's geometry starting from the laws of lever (1978); Spiegel's mean value abacus, designed for the primary grades (1985); Heidenreich's red wine proof of the regularity of the hexagon in the middle of a cube (1987), and many more.

The metaphor of the beam in balance aims at creating one of several suitable basic conceptions (in the sense of Bender, 1991) of the arithmetic mean (see Fig.1). The figure alone, however, does not explain anything. At first, one has to interpret the real line (or rather the interval in question) as a beam without weight (situated horizontally in the field of gravity, which is supposed to be homogeneous), and each of the n numbers a_i, $i = 1, \ldots, n$, as a load with unit weight, fixed to the beam at the position a_i (as long as all numbers are different). In the general case, if the numbers a_i occur with the frequencies k_i ($k_i \in N$), then at each position a_i have to be fixed k_i loads with unit weight, or one load made up of k_i units. This construction gives rise to a fundamental cognitive obstacle that the students have to overcome: the need to interpret the original objects (the numbers a_i) now as positions, and the original attributes (the frequencies k_i) now as objects.

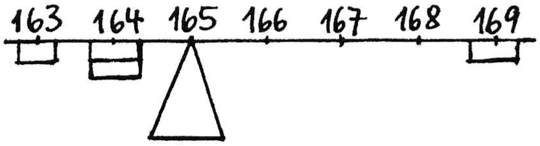

Fig.1: Balance beam

When the beam is supported at a point, it turns into a balance. It is immediately clear that there is one and only one point where the beam is in balance and that this point is somewhere in the "middle." If the left side drops, one has to support the beam at a point farther left to reduce the larger torque on the left and to increase the smaller torque on the right. In fact, one has to go to the left to the point where the two torques are equal. These ideas are plausible even without an elaborate conception of torque and the quantitative laws of lever.

If one knows these laws, one can formulate the facts in the following way: Let $\bar{a}$ be the point of equilibrium, and let m be a natural number ($1 \le m \le n$) so that the positions a_i with $i \le m$ lie to the left of (or include) $\bar{a}$, and those with $i > m$ lie to its right. Then we have $\Sigma_{i \le m} k_i(\bar{a} - a_i) = \Sigma_{i > m} k_i(a_i - \bar{a})$. This equation is well known as description of the equilibrium, and it now has to be recognized as determining a.

If one has the concepts of negative numbers and negative lengths, then the equation simplifies to $\Sigma_i k_i(\bar{a} - a_i) = 0$, with arbitrary numbering of the a_i's. This equation is a characterization of the arithmetic mean $\bar{a}$ (= point of equilibrium) by the fact that the sum of the deviations from a (= sum of the torques) vanishes, that is, that the negative deviations exactly compensate for the positive ones. The contrast to the mean (linear) deviation $\Sigma_i k_i |\bar{a} - a_i|$ is apparent, and there are close relations to other interpretations of the arithmetic mean, given, for example, by the expressions $(\Sigma_i k_i)\bar{a} = \Sigma_i k_i a_i$ and $\bar{a} = \Sigma_i (k_i/\Sigma_j k_j) a_i$.

3.4 This example shows not only that the demand for an active construction of meaning is essential for clarifying the concepts in question but also that, at the same time, it opens new conceptual fields and provokes their exploration. Here again one meets the idea of hermeneutics, which had played an important role already at the beginning of the 19th century. It is no longer simply a normative leading idea of educational thinking, but a suitable theoretical model for describing actual learning processes and thereby supporting that idea.

Thesis 3: Not only are interpretations to convey the allegedly static meanings of mathematical concepts, but they hold new relations and thus contain the germ for new, farther-reaching insights. In addition to their explicatory function, they are provided with an exploratory character that can be utilized for a hermeneutic working style in mathematics instruction as well.

3.5 In the end, visualizations can assume abstract, symbol-like forms. This possibility allows us to treat them like symbols, and correspondingly to see symbols as visualizations (one could call this the *principle of nonvisual intuition*). To evolve their cognitive power, both visualizations and symbols have to be interpreted, although they are grounded in epistemologically different states (Jahnke, 1984). Consider some examples. Discussing the mathematical concept of function, Kiesow and Spallek (1983) emphasize the epistemologic potential of the symbolic mode. Otte (1983) and Fischer (1984a) indicate the intuitive power of the graphical arrangement of formulas. The example we discussed above also supports these ideas. Furthermore, we want to name *matrices* representing linear mappings, *probability spaces* standing for random experiments, and *graphs* embodying abstract relations of any kind. Not merely are these objects symbolizations, formalizations, or schematizations of mathematical ideas, but they demand an active construction of meaning. They contain new structures and promote exploratory ways of doing mathematics.

In these cases, it is the trait of being an object that provides intuitive accessibility. By concentrating on this feature and doing without the sharp contours and consistency attached to real-world facts, one can even make accessible in a nonvisual way such unruly concepts as *infinite point* in projective geometry, *imaginary root* in the theory of equations, and *local rate of change* in calculus (Jahnke, 1989). As these terms suggest, the concepts are to be thought of as extreme cases situated on, or beyond, the boundary of the domain of "ordinary" cases. Their interpretations can be imagined vaguely as limits of the visualizations of ordinary cases, and again, they can be used to shed light on the ordinary cases from a new perspective.