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A logical characterization of the Euclidean plane, sphere and cylinder

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Within a mathemtaical theory, a structured set is called *homogeneous*, if its 'parts' cannot be distinguished from each other by statements of that theory. In Euclidean space geometry, among all convex surfaces and among all complete connected orientable 2-manifolds with at least one regular point, there are only three types of homogeneous objects: the Euclidean plane, sphere and cylinder.

With his famous system of axioms, David Hilbert had given in 1899 a formalistic foundation of (Euclidean) geometry. While, from a mathematical point of view, this axiomatization was fully satisfying, Hilbert's formalistic program as a whole proved to be unrealizable for logical reasons (Gödel 1931). From the beginning, there existed a lot of open questions concerning epistemologic matters, e.g. about the interdependency of the mathematical theory of Euclidean space on the one hand, and real space, together with its artificial and natural geometrical phenomena, on the other hand.

According to Hugo Dingler (1933) [1], geometrical forms can be found in the real world only because man has put them there—by actually making them or, at least, by interpreting shapes, be they man-made or not, in a geometrical manner. By this means, shapes in the real space are conceived as physical approximations of pre-existing geometrical ideas which, again, root in the domain of human needs (in a wide sense). Dingler's philosophy and, in particular, the connection between Euclidean geometry and real world are discussed in detail and are applied to geometry teaching by Bender and Schreiber [2]. There exists a short version of this monograph in English [3], and an outline of Dingler's philosophy, also in English, by Torretti [4].

If one analyses carefully human purposes connected with geometrical forms in social contexts, one comes across with rules for making real objects in real space, i.e. for providing those objects with some useful or esthetic shape in order to fulfil those purposes. Epistemologically speaking, these rules constitute a practical fundament for theoretical norms which generate ideal concepts in Euclidean geometry.

One crucial point in this process is how structures in the physical world are related to logical ones. Paul Lorenzen [5] recognized *distinguishability* (resp. *homogeneity*) to be their elementary common basis, as far as the practical making and the theoretical conception of geometric forms are concerned: Two objects (two parts of an object) can either be distinguished, or not; that is to say *with respect to a given class of means* functioning as touch-stones. A collection of objects (or a single object) is called *homogeneous*, if the objects (its parts) cannot be distinguished;

otherwise it is called *non-homogeneous*. The concept of homogeneity is wide spread in mathematics: topological spaces, factor groups of topological groups, simplicial complexes, polynomials, linear equations, coordinates can be homogeneous. There is always an object whose 'parts' are all of the same kind with respect to some given aspect. In this paper, homogeneity is conceived as a very deep (mathematical and extra-mathematical) idea, a candidate for the list of elements of mathematics in the sense of Halmos [6]. Its meaning is close to that of symmetry.

The idea of homogeneity applies immediately to mathematical sets with their elements, and also, for instance, to surfaces, might they be conceived mathematically as subsets of the Euclidean space or as pre-theoretical objects in real space: A surface is called homogeneous, if its 'parts' cannot be distinguished from each other, e.g. if there are no peaks, no edges, etc. Remember: whether a surface is homogeneous or not, depends on the type of objects to be considered as its 'parts' and on the class of means designated for the test (in the physical world; finger-tips, microscopes, etc., but not, for instance, a device for discriminating colours; in the field of logic: mathematical statements). Making a geometric shape in the real world means working on the surface of some solid, i.e. making it partially homogeneous, with respect to its purpose. The progenitor of all geometric forms which can be met with all kinds of utility articles, tools, machines, etc. is the plane.

Correspondingly, Euclidean geometry can be understood as the elementary theory of homogeneous and non-homogeneous surfaces in the three-dimensional Euclidean space thus representing an idealization of the space surrounding us. Apparently, there seem to be *only three types of homogeneous surfaces*, namely *the sphere*, *the* (unbounded) *circular cylinder*, and, of course, *the plane*. The latter one is excelled among these in that it divides the space into two parts which cannot be distinguished from each other. As the cylinder and the plane are unbounded, they cannot be made in the physical world, i.e. not even approximately. But they can be imagined as surfaces of solids, and they can be made, at least, in parts.

Are there really no more homogeneous surfaces? In order to give a conclusive answer, this question is made accessible to a mathematical analysis by exactifying the notions involved.This formalization of the concept of homogeneity is due to Lorenzen:

Definition. A set S (together with some structure) is called *homogeneous*, if for any two objects x, y and any admissible statement $A(z, S)$ the following conclusion holds:

$$(x \in S \wedge y \in S \wedge A(x, S)) \rightarrow A(y, S) \tag{1}$$

That is, one has to look for admissible statements about the set and its elements, which hold for certain elements and do not hold for others. The set is called homogeneous, if there are no statements of that kind.

The crucial point in this definition is the concept of *admissibility*. For example, let S be a set and $p \in S$ some element (i.e. a constant) and $A(z, S)$ the statement '$z = p$' (where p is used as a parameter). Then S cannot be homogeneous, if it contains more than one element, because for $x \in S$, $x \neq p$ there holds $\neg A(x, S)$, although $p \in S \wedge x \in S \wedge A(p, S)$. If statements like that would be admissible, the concept of homogeneity would be trivial. In order to avoid this trivial case, from now only *parameter-free* statements will be considered.

Of course, homogeneity of a set depends on the structure imposed on that set together with the class of admissible statements. For example, let S be a Euclidean cube (i.e. the surface of a full cube): if statements from differential geometry are admissible, S is not homogeneous as its points can be distinguished, e.g., by the statement 'S is flat in z'. If, however, only statements from general topology are admissible, then S is homogeneous (which we do not prove in this paper).

From now on we consider only *subsets* S of *Euclidean three-space* E (with points z) and *parameter-free statements* $A(z, S)$ from *differential geometry*, including arithmetic, elementary geometry, topology, calculus, etc., as admissible statements.

Candidates for *homogeneous sets* (in this sense) are: E itself; every plane; every plane with one point removed; every plane with one straight line removed; every (unbounded) circular cylinder; every (unbounded) circular cylinder with one circle removed; every sphere; every open half-plane; every (unbounded) cone with its vertex removed; every straight line; every circle; every (regular) helix; every set consisting of the vertices of a regular polygon, including one point sets; the empty set; every union of a plane and all parallel planes having an integer distance from the first one.

Examples for *non-homogeneous sets* are: (full) balls; planes with two points removed; spheres with one point removed; cylinders with one straight line removed; closed half-planes; open disks; cones; non-circular ellipses. Statements by which points in these sets can be distinguished are respectively: 'z lies on the boundary of S', 'z is collinear with two points of the boundary of S', 'z has maximal distance from the boundary of S', 'S has maximal curvature in z', etc.

Obviously, most of these examples and counterexamples are topological subtleties with no equivalent in real-world geometry. There we have only three-dimensional solids with their two-dimensional surfaces being smooth almost everywhere and having no holes. This is our motive to restrict ourselves to subsets of E which are rather good-natured: e.g. for convex surfaces (which are defined as non-empty boundaries of non-empty convex open subsets of E) holds the following theorem:

Theorem A. Let E be the three-dimensional Euclidean space, and let $S \subset E$, $S \neq \emptyset$, be a *convex surface* which is either

(i) *bounded* or
(ii) *connected.*

If S is *homogeneous,* it is either a Euclidean *plane,* or a Euclidean *sphere,* or a Euclidean (unbounded) *circular cylinder.*

Proof. (i) Let S be a *bounded convex surface.* There exists a non-empty bounded convex open subset F of E whose boundary is S, and a uniquely determined minimal closed full Euclidean ball B (with positive diameter) containing $F \cup S$ with at least one point of S lying on the surface S^2 of B. As S is homogeneous, all of its points lie on S^2, that is, $S \subseteq S^2$. Every inner point of B must belong to F, because otherwise there would exist boundary points of F in $B \backslash S^2$, contradicting $S \subseteq S^2$. Hence $B \backslash S^2 = F$ and $S^2 = S$. That is, S is a Euclidean sphere.

(ii) Let S be an *unbounded connected convex surface.* Then S is either a *cylinder,* i.e. the product $C \times L$ of a plane curve C (its basis) and a straight line L, *or not.*

Let $S = C \times L$ be a *cylinder*. S being convex, C must also be convex, i.e. it is the boundary of a convex open subset of the plane, namely either a bounded convex curve, a straight line, a pair of parallel straight lines, or an unbounded curvilinear convex curve.

From the homogeneity of S follows the homogeneity of C, and by an argument as in (i) it follows that, if C is bounded, it must be a circle. Hence S is either a *circular cylinder*, a *plane*, a pair of parallel planes (a case not satisfying connectedness), or a cylinder with an unbounded curvilinear convex basis C.

In order to treat that latter case, we use the notion of *spherical image* $n(C) \subseteq S^1$ of a plane convex curve C, i.e. the set of the endpoints of all unit normal vectors of C directed outwards (where peak points can contribute arcs with positive lengths to $n(C)$). This notion can be applied also to non-convex plane curves, to space curves, to convex surfaces and to two-dimensional manifolds S (whose spherical images lie in S^2), at least when there can be fixed a definite rule which of the two orientations of any normal direction has to be taken, as it is the case with convex surfaces and *orientable* manifolds.

If there is such a rule and every point of C (resp. S) has one and only one normal direction, then there exists the spherical mapping $n: C \to S^1$ (resp. $n: C \to S^2$), which assigns to every point its oriented unit normal. If, for example, C (resp. S) is a manifold and n exists, then n is continuous.

Now let $S = C \times L$ be a cylinder with its basis C being an *unbounded curvilinear convex* curve. $n(C) \subseteq S^1$ is a non-empty (spherically) convex subset of a half-circle containing an open subset of S^1 and also containing its uniquely determined own (spherical) central point. That means, there is an admissible statement $A(z, S)$, namely 'the normal direction in z is the mean one', which is true for some points of S and false for others. So S is not homogeneous, in contradiction to the general assumption.

Finally, let S be an *unbounded connected convex surface*, but *not a cylinder*. According to Stoker [7], in this case the spherical image $n(S)$ is a (spherically) convex subset of a half-sphere of S^2 containing an open subset of S^2. So $n(S)$ contains its uniquely determined (spherical) central point, and the admissible statement 'the normal direction in z is the mean one', again, proves S not to be homogeneous. □

The restriction in theorem A to *convex* surfaces seems to be rather strong. One can drop convexity, if one considers *manifolds* instead of surfaces. As every convex surface is a manifold, theorem A is a direct consequence of theorem B stated below. Nevertheless we treated theorem A separately, without including facts about manifolds, in order to have one more, essentially different, example for the function of homogeneity in proofs.

Theorem B. Let E be the three-dimensional Euclidean space, and let $S \subset E$, $S \neq \varnothing$, be a two-dimensional complete manifold which is either

(i) *bounded* or
(ii) *connected and orientable and has at least one regular point* (i.e. in whose neighbourhood it is of class C^2).

If S is *homogeneous*, it is either a Euclidean *plane*, or a Euclidean *sphere*, or a Euclidean (unbounded) *circular cylinder*.

Remark. By the assumptions are excluded some undesirable cases: lines (not having dimension 2); boundary points not belonging to the manifold like in an open half-plane (contradicting completeness, i.e. allowing Cauchy sequences in S with their limits not in S); two parallel planes (being not connected). Assuming the existence of a regular point is no severe restriction, as, for example, in the case of bounded convex manifolds the non-regular points make up only a set of measure 0.

Proof of theorem B. Let S be homogeneous. As it is complete, it contains its boundary points. So the *boundary* of S is *empty*, because S has inner points. That means that every point of S is an inner one, due to the homogeneity of S.

(i) Let S be a *two-dimensional complete bounded manifold.* Like in theorem A(i), there exists a uniquely determined (Euclidean) sphere S^2 with $S \subseteq S^2$. If the inclusion $S \subset S^2$ would hold, S would have boundary points. So $S = S^2$, i.e. S is a *sphere.*

(ii) Let S be a *two-dimensional orientable complete connected manifold with a regular point.* We already stated that the boundary of S is empty. it also follows immediately that *every* point of S is regular and that S is globally of class C^2.

For every point of S there is defined the Gaussian curvature K, and K is a continuous function $S \to \mathbf{R}$. If $K = 0$ (resp. $K \neq 0$) somewhere, then $K = 0$ (resp. $K \neq 0$) every where, '$K = 0$'(resp. '$K \neq 0$') being an admissible statement, because the equality (resp. inequality) only stands for types of local surfaces: parabolic or planar (resp. elliptic or hyperbolic).

The spherical mapping n (cf. the proof of theorem A(ii)) exists, it is differentiable, and its Jacobian has rank 2 if and only if $K \neq 0$.

If $K \neq 0$, then n is a local homeomorphism for every $z \in S$, and $n(S)$ is an open subset of S^2.

If n is surjective, then it is a covering, and as S^2 is simply connected, S is even globally homeomorphic to S^2. Thus S is bounded and consequently a *sphere.*

If n would not be a surjective mapping onto S^2, there would be points in $n(S)$ with maximal distance from the (non-empty) boundary of $n(S)$ in S^2, and points with non-maximal distance. So S would not be homogeneous.

Now let $K = 0$. As S is complete and without boundary, it is a cylinder $C \times L$ with basis C which needs not to be convex. But C is connected, has no double points, no end points, and in each of its points z it has a well defined curvature k depending continuously on z.

If $k = 0$ somewhere, then $k = 0$ everywhere, and S is a *plane.* So let $k > 0$ (everywhere).

If there are $x, y \in C$ with $x \neq y$ but $k(x) = k(y)$, then k has a local maximum (or minimum) on C; and, C being homogeneous, k has a local maximum (resp. minimum) in *every* point of C. Furthermore, k has a local minimum (resp. maximum) between any two local maximum (resp. minimum) points; so C has at least one local minimum (resp. maximum) point, and hence *every* point of C is also a local minimum (resp. maximum) point. So k must be locally constant everywhere, and, being continuous, it must be constant at all. So C is a *circle* with radius $1/k$.

Now let k be strictly monotone, hence C not closed. For every $z \in C$ the rest of C (in the direction of increasing k) is contained in a disk with radius $1/k(z)$, and, being complete, C has a boundary point. Thus C and, consequently, S are not homogeneous, in contradiction to the general assumption. $\square$

Remarks. (i) We did not prove that the plane, the sphere, or the cylinder really *are* homogeneous. A proof of this fact needs means basically different from those used above, namely logic induction on elementary statements.

(ii) The plane has got some higher 'degree of homogeneity' than the sphere and the circular cylinder: It is not only homogeneous itself, but also its complement consists of two undistinguishable half-spaces and is obviously homogeneous in the sense of the definition given above. The complements of the sphere and the cylinder, on the contrary, are not homogeneous. This can be seen with the help of the statement $A(z, E\backslash S)$ = 'there is a plane through z completely contained in $E\backslash S$'.

(iii) There are no homogeneous connected lines in E besides the straight line, the circle and the (regular) helix.

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