Department of Teacher Education

Mathematics Teaching from a Constructivist Point of View

Ole Björkqvist (red.)

Reports from the Faculty of Education, Åbo Akademi University

No. 3 1998

ISBN 952-12-0151-7
ISSN 1236-0856

Basic Imagery and Understandings for Mathematical Concepts

Peter Bender
Universität-GH Paderborn, Germany

An overwhelming part of everyday mathematics teaching all over the world can be characterized by the absence of meaning for the students. The didactical conception of basic imagery and understandings (BIU) is an attempt to make mathematics teaching meaningful. This conception takes into account the ambiguity of mathematical concepts in the realms of subject matter, of epistemology and of psychology and the crucial problem of matching the concepts from different realms referring to the 'same' object. The underlying epistemology is that of social constructivism, however ascribing a predominant role to the teacher and her or his utterances.

The contribution of epistemological constructivism to mathematics education

Modern mathematics didactics is confronted with at least two profound epistemological problems which are logically independent from, but which are practically closely connected with, each other: First, there is the basic, but by no means trivial, relationship between those concepts and contents belonging to the more or less pure sphere of the discipline of mathematics, those concepts and contents which we educators want the students to acquire, and those concepts and contents which the students really cognitively construct (or, better: which we suppose the students to construct). Second, there is the fundamental question of how learners acquire knowledge and form concepts, which can be embedded in the more general question of how individuals interact with the (cognitive) world around them. By asking this question in this way I aim at the epistemological philosophy of constructivism, after which "the function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality" (von Glasersfeld 1989, 182).

Although my focus is on the first of these two problems, namely the ontological relationship between the realms of subject matters, epistemology and psychology, there is, throughout the paper, a continuous noise in the background, caused by constructivist ideas. – The extreme variant of constructivism denies the existence of commonly shared knowledge and concepts at all, and there arise fundamental difficulties, e.g. how to explain the broad, surely low, but nevertheless existing, success of school instruction, which shows in obviously largely commonly shared knowledge and concepts (despite billions of well known and sometimes well documented failures). In order to overcome those difficulties, there have been developed several modifications of radical constructivism (in the sense of its protagonist, von Glasersfeld). One of the most interesting variants seems to be *social constructivism* (in the sense of Ernest, 1994a), which postulates the pre-existence of social reality to every individual, the generative character of human conversation, and the possibility of collective mental functioning.

It is doubtful whether social constructivism actually is constructivism, as in terms of radical constructivism the social reality is part of the individual's construction of the world, and conversations, as well as seemingly collective mental processes have no objective character at all, but belong to each individual's experiential world which she or he organizes in a subjective way.

My ideas of how to bridge the gap between subject matters, epistemology and psychology are closely connected with the so called social constructivism, as can be seen in the following chapters. – But surprisingly, from a radical constructivist point of view there have to be drawn similar instructional consequences (which within this philosophy would not be called 'instructional'): It is the teacher's profession to offer the students possibilities for constructing their knowledge and concepts. Although, on principle, the teacher cannot control or fully understand the students' constructions, she or he can judge to some extent – by observing how the students talk about their knowledge and their concepts, how they apply them or how they build theories with their help –, whether the students' constructions match her or his own knowledge and concepts (which she or he has proved to be adequate in some way by mental reflections, in conversations with other people, including students, applications and formations of theories – permanently due to more or less slight changes). In this

sense, good instruction is nothing else but the *preparation of (relatively) relevant settings* as realistic, as well as possibly artificial, parts of the students' experiential worlds, in order to stimulate their construction of knowledge and concepts.

What is more: If there actually is no direct transfer of knowledge and concepts, all the more teachers have to develop 'instructional' settings with particular care, taking into account not only subject matters, but also their students' personalities, as well as social, pedagogical etc. variables. Even if, on principle, teachers will never really know about their 'success', they are obliged to strive for arranging optimal (in whatever sense) settings. – All the implications for teaching practice which are readily drawn from constructivist perspectives, like

- sensitivity towards the learner's previous constructions,
- diagnostic teaching with cognitive conflict techniques for remedying learner errors and misconceptions,
- attention to self-regulation by learners,
- use of multiple representations,
- concern with learner cognitions as well as with teacher cognitions,
- problematizing knowledge as a whole,
- taking into account that there is no 'royal road' to truth,
- awareness of the dichotomy between learners and teacher goals,
- awareness of the linguistic basis and the social construction of mathematical knowledge,

etc. (cf. Ernest 1994b, 338f) are well coherent with my plea for careful preparation of instructional settings. Compared with this, in my opinion, the "progressive teaching ideology [which] sanctions anything the child does as expressions of its individual creativity, and naively assumes that the child can discover much of conventional school knowledge on its own" (Ernest 1994b, 337) is, if based on the philosophy of constructivism, a misunderstanding.

I will now turn to my proper subject, i.e. "basic imagery and understandings for mathematical concepts" which constitutes a substantiation of (moderate) constructivist ideas, structured along the key words 'meaning', 'basic', 'imagery' and 'understandings', explained along concrete mathematical examples, and now and then enriched with the discussion of constructivist arguments.

The problem of judging the 'adequacy' of students' 'concepts'

Despite immense progress in the field of mathematics didactics there are still a lot of mathematics educators as well as teachers who adhere to a rather narrow picture of their subject, namely consisting on the whole of abstract relations between abstract objects and some calculation. For them, intuitive, vivid, enactive or application oriented ways of doing mathematics do not belong to true mathematics, but are mere approaches. The advantage of this picture is that the contents can be identified exactly, and can easily be made accessible to presentations in textbooks, as well as to empirical research on how students handle them, or to (so called) intelligent tutorial systems.

Yet, this picture is not a suitable foundation for teaching and learning mathematics (neither for doing or applying mathematics), as in it the category of *meaning* is ignored and hence the constitution of meaning is not a matter of education. – But in every thinking or learning process the individual assigns some meaning to some notion, situation or circumstances, and teachers, in particular mathematics teachers, have to take into account these processes of assignment.

Closely connected to the difficulties in recognizing and controlling the students' learning processes is the problem of matching concepts in the realms of mathematics, epistemology and psychology (which I will call 'mathematical', 'epistemological' and 'psychological concepts' respectively). – The conception of basic imagery and understandings (BIU) offers a didactical frame for this matching problem. In German mathematics education this conception has a long tradition. Rudolf vom Hofe (1995) investigated its history and found a lot of variants in the last 200 years, most of them tackling the matching problem by designing ideal normative mathematical concepts in the epistemological mode (vom Hofe names them "basic ideas") serving as models for the students' formation of concepts in the cognitive mode (which he names "individual images").

It seemed to be natural to all those educators to found their conceptions on an analysis of subject matter and to include their rich teaching experience as an empirical background. Thus they were much closer to their students than many mathematics professors at the universities or teachers at the Gymnasiums (in

former times with the top 10 % of each age-group) who taught (and often still do teach) mathematics – maybe in an elementarized, but still – in a rarely modified manner as a pure discipline. On the other hand those educators, too, did often not care for what really happens in the students' brains, and, furthermore, in spite of their good ideas their efforts had only success up to a modest level.

But one must admit that only since the 1970s there has been reasonable technology for thoroughly studying classroom actions, namely video recordings. Of course, even with this technology one does still not know how cognition 'really' works. Neither the mathematical formalization of thinking processes, nor the definition of man as an information processing being similar to a computer (Simon 1969) brought about much new insight in *human* cognition. But based on the talents of video technology we learned a lot about communicative and social interaction in the classroom, in particular, how mathematical meaning is implicitly and explicitly negotiated between the participants (cf. Bishop 1985).

Due to the constructivist and connectionist roots of their theories, some cognitive scientists underestimate, ignore or deny a dominant influence of the teacher and, consequently, of the subject matter on the students' learning processes. – In fact, during painstaking examinations of video taped and transcribed micro situations middle and long term effects can easily get out of sight. If one concentrates on social and communicational characteristics of a situation, the subject matter tends to play only a minor role. And comparing students' deviating verbal and non-verbal manifestations with teachers' obvious original intentions may support severe doubts in the efficacy (or even possibility) of extraneously determined learning processes. – These tendencies are supported by the researchers' aim to overcome the old theories because of their meager success.

On the other hand careful re-analyses of classroom situations under *subject matter aspects* often lead to plausible recasts or improvements (as well as to verifications) of former interpretations based on interaction-theoretical grounds. So to me it sounds unreasonable to exclude these aspects when exploring such a situation. As I pointed out before, in German mathematics didactics, for a lot of mathematical concepts there are well known elaborated teaching routines. Whether a teacher relies on such a routine or not: From the words, diagrams etc. that she or he uses, from her or his rejection or acceptance of students' answers etc., the observer frequently can disclose the teacher's own imagery and understandings about the mathematical concept in question. (Throughout this paper, the notion of mathematical concept includes theorems, mathematical structures, procedures etc.) – Surprisingly often, the teacher's own imagery and understandings seem to be inadequate, or at least the teacher evokes inadequate imagery and understandings among the students. – These problems can be tackled didactically with the help of the conception of BIU, which is meant to be a theoretical and practical frame for a normative, descriptive and constructive treatment of concept formation processes.

A radical constructivist would argue that there is no adequacy or inadequacy of imagery and understandings. Here we have reached a point of discourse where there might be no agreement. For me, *adequacy* of concepts or adequacy of imagery and understandings is a useful and important didactical category. Of course, adequacy cannot be proved like a mathematical theorem. Whether a student's concept is adequate even cannot be stated uniquely, neither in the prescriptive, nor in the descriptive mode. But there are strong hints in either mode: If a student's statements about, and actions with a mathematical concept sound plausible and seem to be successful to her or his own common sense as well as to experts, we would concede some adequacy (for more details cf. E.J. Davis 1978).

As stated in the beginning – from a didactical point of view, it is not crucial whether teachers actually 'teach' their students or whether they only *stimulate* their students' concept formation processes. Good 'teaching' always contains stimulating the students' own acitivities.

Explanation of the key word 'basic'

By the adjective 'basic' there are expressed several essential characteristics of the conception of BIU:

- It includes a tendency of *epistemological homogeneousness and obligation* how mathematical concepts should be understood.
- *Psychologically* speaking, it indicates that students' individual concepts normally are, and in the teaching processes the epistemological concepts should be, *anchored in* the students' *worlds of experience.*

- With respect to *subject matter* it stresses the importance of *fundamental ideas* (in the sense of Halmos' elements, 1981, or Schreiber's universal ideas, 1983) guiding the study of any mathematical discipline.

Epistemological homogeneousness: This tendency seems to be in contradiction with modern pedagogical and didactical paradigms like "the students should create their own mathematics", or "the students have to find their individual ways in solving mathematical problems" etc. In fact, teaching does not mean telling (Campbell and Dawson, 1995), but it means stimulating students' cognitive activities, negotiating mathematical meaning in the classroom etc.

But this way of conceiving the teaching-learning process does not entail any obligation for the teacher to tolerate or even to support inadequate individual concepts, on the contrary: it makes the teacher's task much more difficult. She or he must be provided with a good theoretical and practical competency in mathematics, mathematics applications, epistemology, pedagogy, psychology, social sciences etc., in order to

- develop her or his own view of the epistemological kernel (which must not be identified with a mathematical definition) of some mathematical concept which the students shall acquire,
- perceive the students' actual individual concepts as truly as possible and judge their adequacy,
- help the students, if necessary, to improve or to correct their individual concepts into adequate ones near the epistemological kernel,
- possibly learn by the students and improve her or his own individual concepts.

This task imparts a predominant role in the teaching-learning process to the teacher's own imagery and understandings and to their transposition into didactical action. For example, if for the calculation of π a circle is approximated by a sequence of polygons and the teacher uses a phrase like "in this sequence the polygons have more and more vertices, and finally they turn into the circle", the students' formation of an adequate concept of limit is obstructed.

The epistemological kernel of a concept corresponds to a commonly shared socio-psychological kernel. Such a socially constituted kernel is an important prerequisite for the construction of individual argumentation and its introduction again into classroom interaction (cf. Krummheuer, 1989). – It is obvious that this commonly shared kernel should be as extensive as possible, which, again, gives the teacher a central position in the teaching-learning process.

Anchoring mathematical concepts in the students' worlds of experience: Even working mathematicians need some real world frame for doing mathematics ("we consider ...", "if x runs through the real line ..." etc.; cf. Kaput, 1979, and many others). All the more students need such frames so that they can constitute meaning with the subject matter they are about to learn (Davis and McKnight, 1980; Johnson, 1987; Fischbein, 1987, 1989; Dörfler 1996 etc.). As such frames do not belong to the epistemological concepts, the teacher is rather free when constructing real world situations where basic imagery and understandings can be unfolded.

These situations need not be absolutely realistic, on the contrary, by alienating them with the help of fairy-tale traits and concentrating on the essence they can be turned into metaphors with their explanatory power. One can take human beings, animals, things, which are more or less anthropomorphized and more or less mathematized. These participants of the situation have to act somehow, following some arbitrary rules, pursuing some arbitrary plans, obeying arbitrarily physical and other natural laws, or not.

For a lesson about the integral as area function for a given function I designed the following situation: The x-axis is a hard-surface road; north of this road (in the coordinate system) there is a uniformly wet swamp which is bounded by the road and the graph of the function in question. A vehicle drives on the road in positive (eastern) direction, with an arm perpendicular to the road which is sufficiently long to reach all parts of the swamp during the trip. With the help of this arm the water is *absorbed* uniformly from the swamp (on the basis of some uninteresting technology) and collected in a cylindrical jar. Thus, at any moment the level of the water in the jar is a linear measure of that part of the area which has already been passed by the vehicle.

If the vehicle reaches a position where the function is negative, the metaphor has to be extended: South of the road there is a desert which is bounded by the road and the negative parts of the graph

and which has to be *watered* uniformly by the vehicle. For this purpose the vehicle has a second arm perpendicular to the road which is sufficiently long to reach all parts of the desert during the trip. Again, the exact mechanism is not interesting; the only important thing is that the level of the water in the jar drops proportionally with the desert area passed.

Of course, this metaphor contains a lot of technical and didactical problems which have to be considered thoroughly: – What happens if the jar is full (empty) and there is still swamp (desert) area to be drained (watered)? – Draining the swamp and watering the desert have to be accomplished with the same velocity of flow (whatever this physical notion means). – On principle, one needs a new co-ordinate system for the function of the water level (the integral function). – When the vehicle makes a half turn and then drives in the negative (western) direction, the two arms change their positions, and now the desert has to be in the north and the swamp has to be in the south of the road (in accordance with the mathematical changing of positive and negative area). – But if the starting point of the vehicle is finally made a variable, the efficiency of the metaphor comes to an end.

Every metaphor has its limitations (cf. Presmeg, 1994), but this is no drawback. The one which I just described should make plausible

- continuous measurement,
- the transfer from area measurement into linear measurement and
- the concept of negative area.

It thus appeals to common sense, and if the teacher wants the students to maintain their common sense, it is a must to emphasize the limitations of any metaphor.

Situations which are appropriate for mathematics teaching rarely come along by themselves. Genuine mathematics applications are often not suited for supporting concept formation, as they are frequently overloaded with alien problems. At the same time the teacher should not evoke the impression that some artificial situation, designed for the use in mathematics teaching, would be an example for genuine mathematics applications. Sometimes, this coincidence can happen, but usually it does not; and students with common sense realize the artificial character of such a situation.

Fundamental ideas for mathematical disciplines (in an epistemological and psychological sense): Basic imagery and understandings are not only meant as a peg on which to hang some mathematical content, but they shall lay the foundations for further meaningful interpretations of concepts within a mathematical discipline.

Imagery and understandings

The notions of imagery and understanding stand for two fundamental psychological constructs. There exists an extensive literature about them. Different authors have different definitions, most of them not very concise. A lot of contemporary cognitive scientists disregard these two constructs anyway, as they escape hard empirical research and do not fit a computer related view of intelligence. – But it is just these shortcomings (seen behav-iouristically), their vagueness and flexibility, which turn these constructs into suitable means for analyzing (and promoting) such complex didactical objects as human teaching-learning processes.

Imagery can be grasped as: mental, often visual (but also auditory, olfactory, tactile, gustatory and kinesthetic; cf. Sheehan, 1972) representations of some object, situation, action etc. having their sensory foundations in the long term memory and being activated in *conscious* processes. A person activating some imagery has already some meaning, some intentions in mind and organizes these processes according to these intentions (Bosshardt, 1981). – Imagery is closely related to intuitions, but its objects are more concrete, and meaning plays a more important role.

The objects of imagery (and understandings) can be given in different modes, namely analogous or propositional. I don't want to resume the cognitive scientists' quarrel in the 1970s about the interrelations between these two modes or about their separate existence as ways of thinking. In my opinion both are valuable means for analyzing imagery and understandings in teaching-learning processes.

Apparently, imagery is more closely connected to the analogous mode, and understandings are more closely connected to the propositional mode of thinking. But it is difficult for a person to activate some imagery without propositional elements, in particular in didactical situations, as in these situations

verbalization is *the* fundamental means for a participant to communicate either with others or with her- or himself (this communication with oneself being a transposition of a social situation to one's mind which is typical for teaching-learning processes). On the other hand, there can be no process of understanding without recurring to any plausible imagery and to analogous elements.

Obviously, thinking in the analogous mode can be stimulated by analogous means like pictures, diagrams etc. (with a lot of limitations; cf. Presmeg, 1994), and the propositional mode can rather be stimulated by propositional means like verbal communication. In the age of paper and pencil and of books, analogously given objects frequently are of a visual, static nature, and the learners have to undertake some effort to make these 'objects' plausible, meaningful, vivid imagery matching their worlds of experience. In the nearest future, the use of multimedia in schools (in the western world) possibly will relieve the students from these efforts.

Whether multimedia will be conducive to the students' learning processes, is not yet settled: The students' inclinations and abilities to undertake efforts to generate mathematical concepts could be undermined. – This problem is complementary to the following classical one, related to the use of visualizations (diagrams, icons etc.): Among educators there is a naive belief that visualizations do facilitate the students' learning processes. But as, for example, Schipper (1982) showed with primary graders, many visualizations are not self-explanatory at all, but they are subject matter which has to be acquired for its own sake on the one hand, and in relation to the visualized contents on the other hand. – As a matter of course, visualizations can be successful didactical means, but not because they would reduce necessary effort, but because they demand more effort and give hints how to direct and structure this surplus effort and thus make it effective.

There are didactical situations, as well as mathematical concepts, as well as students, for which resp. for whom one of both modes is more suitable. For teaching and learning mathematics it is important that there has to be a permanent transformation between the two modes. Maybe geometry can be treated predominantly in the analogous mode, and algebra in the propositional mode, maybe the teacher is even able to take into consideration the preferences of single students. But on principle, both modes must be present.

Taking into account the wide-spread propositional appearance of mathematics teaching, in particular on the secondary level, there is need of an increased use of the analogous mode all over the world. – By stressing the students' anchoring of their individual concepts in their worlds of experience, the conception of BIU lays some accent on the analogous mode, as a prophylactic counterweight to the preponderance of the propositional mode in the upper mathematical curriculum.

The psychological construct of *understanding* is still more complicated, non-uniform and, from a constructivist point of view, questionable. For didactical reasons the following aspects are relevant:

(1) One can understand people, their actions, situations, the motives or the aims of the participants (practical knowledge of human nature, *common sense*).

(2) One can understand utterances *medially* and *formally* (e.g., if they are made loud enough and in a language one knows).

(3) One can understand the *content* of a *message* made by someone (understand what this someone *means* by a certain communication, text, phrase, word, symbol, drawing etc.).

(4) One can understand technical matters, working principles of gadgets, mathematical structures, procedures etc. (*expertise*).

At first glance, aspect (4) seems to be most suitable for the conception of BIU. But it becomes immediately clear that each of these aspects is important for the learning of mathematics and has to play an essential role in the conception, in particular (3). This aspect is a classical psychological paradigm, but the general opinion about it has changed, not least under the influence of constructivism: Today, one does not believe anymore that it consists just of finding some objective meaning of given signs, but that the receiver of a message tends to and has to embed the message in some context and, in doing so, tries to reconstruct its meaning (cf. Engelkamp, 1984), thus getting near aspect (1).

It goes without saying that there is no understanding (3) without (2): the sender and the receiver of a message have to have a common language, not only in a direct, but also in a figurative sense: As Clark and Carlson (1981) put it, there has to be a "common ground", which, again, refers to aspect (1). – In school teaching, and in particular in mathematics teaching, the common ground of teachers and students is often rather thin, if existing at all. – But, extending the common ground does not only mean that the students have to be better instructed so that they make the teachers ground their own. Rather the teacher must engage in the students, attach importance to them (and not only to the subject matter), understand them as human beings (again, aspect (1)), and try to reconstruct or to anticipate their ways of thinking.

By following the conception of BIU, to some extent the teacher is *forced* to do so, and furthermore, her or his expertise can be promoted. But this way of teaching and learning demands much more effort for both parts, in comparison with the ususal way, where teachers, in good harmony with the students, are satisfied with students' instrumental understanding (in the sense of the late Robert Skemp, 1976).

In the following example, the teacher (resp. the researcher) did not quite understand the student's ways of thinking. It was originally described by Malle (1988) and re-analyzed by vom Hofe (1996):

In order to develop the concept of negative numbers, Ingo, the student, was given the following situation: "In the evening the temperature is 5 degrees (Celsius) below zero. During the night a warm wind moves inland, and the temperature rises *by* 12 degrees. – What is the temperature next morning?" Ingo answers correctly: "7 degrees", but in the dialogue with the interviewer, he shows inadequate imagery. When he shall sketch a picture of the situation, he asks whether he must draw three thermometers, and later he explains that at midnight the temperature went up to +12 degrees, and in the morning it dropped to +7 degrees.

Malle gives well known and, of course, correct explanations for Ingo's obvious inadequate dealing with the situation: Ingo is not able to identify the elements which are important for solving the problem, but invents additional information and tells fairy tales, and he does not differ between the starting and the final state (i.e. the starting and final temperature, represented on the thermometer) on the one hand and the change between the states (the rise of the temperature) on the other hand.

In his careful re-analysis, vom Hofe shows that the problem lies in Ingo's imagery about the physical situation, which is no suitable basis for the formation of the mathematical concept. Whereas the interviewer expects Ingo to focus on the changes of the mercury column (as a direct model of the number line), Ingo imagines two masses of air, a cold and a warm one, which mix and result in a third mass with average temperature. Therefore he needs *three* thermometers, and in the night the temperature does not rise *by* 12 degrees, but *up to* 12 degrees, and goes down again in the morning. The idea of mixing air masses is, physically speaking, not at all inept, but it merely does not fit the mathematics that the interviewer has in his mind. For Ingo, there are two states of temperature which result in a third one, the weighed arithmetical mean, and not one state which changes into another.

Granted that every human being tends continually to conceive, or to make and to keep her or his environment meaningful and sensible, one must admit that usual mathematics teaching in large parts has a contra-productive effect. – The strive for "constance of meaning" (Hörmann, 1976) is in my opinion a characteristic trait of humans, which, for example, is largely ignored in Piaget's biologistic theory of equilibration.

Mathematics teaching, too, is such an environment which humans who are in touch with it try to make meaningful and sensible. As an *extreme* example, (in a famous French movie from 1984) in a physics lesson in Paris the absent-minded student from Algeria understands "le thé au harème d'Archimède", when he hears "le théorème d'Archimède" (which means: "tea time in Archimedes's harem" instead of "the theorem of Archimedes"). Even if we omit such extreme cases, it still seems to be rather normal all over the world that students tend to develop their own non-conformistic imagery and understandings, which, however, often remain implicit.

Fischbein (1989) calls them "tacit models" and characterizes them as simple, concrete, practical, behavioural, robust, autonomous and narrowing. Their robustness results from their simplicity, their anchoring in the students' worlds of experience, and their short term success with convenient applications (see the example of Ingo

and the temperature). Inadequate tacit models come into being because of lack of adequate basic imagery and understandings, which in their turn would also be concrete, practical, etc., successful and therefore robust, and not narrowing, but capable of expansion. So the conception of BIU includes the strategy of occupying the students' frames with adequate basic imagery and understandings from the beginning, i.e. to give them the possibility and to enable them to develop such imagery and understandings by themselves.

Nevertheless, students will still generate a lot of inadequate tacit models, and teachers must be able to recognize them and to help the students to settle them. In this, again, the teachers can be supported by the theoretical and practical frame of the conception of BIU, thus using the constructive aspect of the conception (as vom Hofe, 1995, puts it).

Fischbein (1989), like many other educators and cognitive scientists, recommends that the students should undertake meta-cognitive analyses in order to discover and eliminate the defects in their frames. – I couldn't find evidence in the literature that students would be able to successfully analyze their own (wrong) thinking without massive interventions by the teacher or by some interviewer. According to my own experience with young people in all grades, they are overstrained if they shall reflect reflexively about their own reflections.

Indeed, in many classroom situations there can be found actions of understanding on a meta-level; for example, if students recall how they solved a certain problem, or if they try to find out the teacher's intentions, instead of trying to understand the contents of her or his statements. But, in general, this kind of understanding (aspect (1)) is not explicitly reflected by the students.

One essential trait of every didactical situation is (or should be) that the participants strive for understanding the contents of some message given, verbally or non-verbally, by the teacher, students, the textbook etc. (aspect (3)), with the underlying aim that the students shall acquire expertise (aspect (4)). Whereas aspect (3) stresses the *processes* of understanding, aspect (4) stands for the *products* of these processes. The products are not only results, but at the same time they are starting points for new processes, and each understanding process starts on the ground of some already existing understanding.

In mathematics teaching, both aspects of understanding ((3) and (4)) **deal with the same objects**: the messages, seen ideally, deal **with mathematical** concepts, about which the students shall acquire **expertise.** – In the humanities and in the social sciences, as well as **(in an indirect way)** in mathematics, this expertise again often **refers** to social situations (in a wide sense) and thus is in parts **identical** with aspect (1). – So, finally, in normal teaching-learning **processes** all the aspects of understanding discussed here belong **together** and are essential for success.

In my view, there is no understanding without imagery, and no imagery without understanding. With the notion of imagery there are stressed the analogous mode, roots in everyday lives, intuitions etc., – whereas with the notion of understanding there is laid some accent on the propositional mode, on subject matter, on predicates etc., – but both notions do not only appear together, rather they have a large domain of essence in common.

Examples from mathematics

There can be identified roughly four types of BIU for the use in mathematics teaching in the primary and secondary grades:

A. More or less *global* BIU, especially for the formation of the concept of number and for elementary arithmetic: multiplication as repeated addition; division as partitioning (splitting up; 'Aufteilen') or distributing (sharing out; 'Verteilen'); fractions as quantities or as operators, negative numbers as states or as operators, the machine model for operators, the little-people metaphor for running through an algorithm. Basic imagery and understandings are not bound to primitive, non-quantifiable actions (in the sense of intuitive understanding according to Herscovics and Bergeron, 1983), and their formation is not a kind of mathematical propedeutics or pre-mathematics, but – in my opinion – genuine mathematics (just without calculus with symbols). They would be useful in the upper secondary grades as well, for example with the concept of limit and infinitesimal thinking as a whole.

B. More or less *local* BIU, e.g. the arithmetical mean, the internal rate of return of an investment, the circumcircle of a regular polygon.

C. BIU for *extra-mathematical* concepts, situations, procedures (from physics, economics, everyday lives etc.), which are to be used in mathematics teaching (example: Ingo and the temperature).

D. BIU for *conventions*, e.g. the meaning of symbols, or of diagrams. Example: The teacher tries to explain subtraction with the help of the following situation: "Mother baked six cakes for her daughter's birthday; the dog Schnucki ate four of them. How many are there left?" She draws 6 circles on the blackboard and crosses out four of them (each with one line), hoping to support visually the understanding of the problem 6–4=2: But Ralph, a learning disabled child, wonders why the teacher halves the marbles (Mann 1991).

It goes without saying that the prototypes, metaphors, metonymies (Presmeg 1994) used for BIU should not obscure the concepts they refer to, like in the following example:

Euclidean geometry: As a preparation for proving the existence and uniqueness of the incircle of a triangle, the teacher asks the students: "Imagine a cone with several balls of ice cream intersecting each other physically, and a plane section through the cone containing its axis of rotation. Do so three times, by identifying the tip of the cone with the three vertices of the triangle one after another." – A more suitable imagery would be: to stick a small circle near one vertex between its adjacent edges. When the circle is blown up like a two-dimensional balloon, it moves away from that vertex, still touching the two edges, until it meets the third edge and thus reaches its final position as incircle. In dissociation from the pure Euclidean way of doing geometry, this metaphor makes use of kinematic and continuous physical phenomena from the students' worlds of experience.

Of course, mathematical concepts should not be falsified, as it is the case with the concept of circle in Papert's (1980) original idea of *turtle geometry*. The children shall draw a circle by programming the turtle to do a straight motion of length 1, then to do a right turn of the amount 1, and to repeat these two actions 360 times, i.e. by drawing a fuzzy regular polygon with 360 vertices. In fact, the result looks like a circle, but the way in which it was produced belongs to a concept which is essentially different from the Euclidean circle. It's true, that every line on the computer screen is a sequence of squares; but this is not the point, as students with some experience with paper and pencil as well as with computer screens will recognize the shortcomings of any realization of geometric forms and will be able to idealize these forms, if at least the underlying activities are appropriate. – But the procedure for making a Logo circle is not appropriate for Euclidean geometry. – Furthermore, I doubt that the Logo geometry is a good preparation for differential geometry – eventually it is a helpful model for someone who already has the concept of mathematical limit at her or his disposal, whereas it is likely to be a mental obstacle for someone who is still on her or his way of acquiring this concept, let alone for primary graders.

Transformation geometry: When in the late 1960s and early 1970s transformation geometry was pushed into the mathematics curriculum, it was assumed that real motions of real objects could serve as BIU. In fact, the students accepted these BIU willingly and transferred them easily into continuous motions of point sets in the plane. But the crucial point was the abstraction of the motions, which the students in general did not manage to perform. Their BIU of transformations grounded on motions were so robust that the good advice to focus their attention to the starting and final positions of the geometric forms or to the plane as a whole remained useless, because, for example, the notions of starting and final position, again, evoked imagery about motions (cf. Bender, 1982).

Thus, mathematicians and mathematics educators failed to establish in the curriculum the full algebraization of geometry by transformation groups, and up to today geometric transformations are not treated as objects on their own, but only as means to investigate geometric forms. The idea of embodying Piaget's groupings of thinking schemes in geometric transformation groups had proved to be too naive.

By the way, there are reasonable didactical applications of continuous motions, e.g. in good old *congruence geometry* by Euclid and Hilbert: Two geometric forms are conguent, if they can be moved to each other in a way that both exactly cover one another. In German there is a synonym for the word 'kongruent' which is due to this reciprocal covering (='decken'), namely 'deckungsgleich' ('gleich'= 'equal'). In congruence geometry, different from transformation geometry, the specific form of these

motions is not essential at all. So the students need not, cannot and, in fact, do not memorize them, and motions are not likely to turn into mental obstacles against viewing congruence as an interrelation between two stationary geometrical forms.

For *functional reasoning* in geometry and other mathematical disciplines, like calculus, there is needed a different, and slightly more abstract, concept of motion: What happens in the range of a function, if one 'walks' around in its domain? Example: The area function assigns to each triangle of the Euclidean plane its area. Starting with one triangle, one changes one of its vertices, and one observes, how the area changes. – The metaphorical character of this situation is obvious: There is a space (the domain, i.e. the set of all triangles) and someone or something (the variables) who 'walks' around; and the motions of this someone or something are transferred by some mechanism, like an abstract pantograph, into another space (the range, i.e. the positive real numbers).

One more example, where 'dynamic' imagery and understandings seem to be not helpful for basic concept formation, is the *concept of sequence and limit:* Many students have the wrong idea that a mathematical sequence would possess a last element (with the number ∞) or that one could at least reach such an element (whatever the notion of reaching should mean). – The ground for this misconception is often laid in mathematics teaching itself, e.g. when determining the number π by an approximation:

The students consider a sequence of polygons which have more and more vertices until they finally turn into a circle. Even if the teacher carefully avoids such a wrong diction, the students still can easily get the impression that the circle would be the last element of that sequence: Firstly, because of optical reasons, and secondly, because the aim of the lesson *is* to determine a limit by a sequence of elements, and all the activities evoke this impression, whether the teacher expresses verbally that the limit cannot be reached, or not. Even if the students accept that it is impossible to reach it, they tend to ground the impossibility on limited time and limited arithmetic of human or electronic calculators.

Another example, which is dealt with in the curriculum even earlier, is the decimal fractions of *rational numbers.* The students prove, e.g., that $\frac{1}{3} = 0{,}333...$. The teacher states that the equality holds if there are infinitely many digits '3', and finally, in Germany, there is written $\frac{1}{3} = 0,\overline{3}$ as abbreviation. By this notion the double nature of the concept of limit is expressed. The symbol 'lim...' stands for a request to run through a process and, at the same time, for the result of this process.

For an *algebraic* term, like a+b , this double nature (to be a request for some activity and to be the result of this activity) is well known and useful, but it fails when the activity includes some infinite process. So the students are right, when they refuse to accept the correctness of the equality $0,\overline{3} = \frac{1}{3}$ and all the more $0,\overline{9} = 1$. They take the dynamic part of the double nature of limit seriously (because this part, grounded on the didactical principle of supporting 'dynamical' thinking, is always stressed), and they correctly deny that running through that infinite process, for which an expression like $0,\overline{9}$ stands, will result in the limit. Fischbein (1989) observed that students even deny the symmetry of the equality sign, as they accept $\frac{1}{3} = 0,\overline{3}$, because this expression can be read from left to right, and the digits on the right can be written down one after the other, whereas they refuse $0,\overline{3} = \frac{1}{3}$, because one can never have on the left side all the needed ingredients to produce the result, one never comes to an end and one is not able to say " $= \frac{1}{3}$ ".

The place of the conception of BIU in mathematics didactics

In all times, all over the world, mathematics educators reflected and still do reflect on basic imagery and understandings for mathematical concepts, though they usually do not name them like that and possibly have different or no conceptual frames. There is still missing a theory unifying the relevant disciplines 'mathematics', 'epistemology' and 'psychology'. The work of vom Hofe and my work is one attempt. But the realization in didactical and teaching practice is at least as important as the theory. Which basic imagery and understandings do we think to be adequate? How can we support the students generating adequate basic imagery and understandings? Which inadequate basic imagery and understandings can occur? How are they caused? How can they be improved or corrected? – In my opinion, these are fundamental

questions of mathematics education. These questions, as well as my attempt to give preliminary and local answers, are closely connected with Ernest's (1994a) ideas of social constructivism, but in my opinion they are also compatible with more radical variants of constructivism.

References

Bender, P. (1982). Abbildungsgeometrie in der didaktischen Diskussion. *Zentralblatt für Didaktik der Mathematik 14*, 9-24.

Bishop, A. J. (1985). The Social Construction of Meaning – A Significant Development for Mathematics Education. *For the Learning of Mathematics 5* (1), 24-28.

Bosshardt, H.–G. (1981). Vorstellungen – Grenzgänger zwischen Wissenschaft und Alltagsverstand. *Zeitschrift für Semiotik 3*, 363-370.

Campbell, S. and Dawson, A.J. (1995). Learning as Embodied Action. In R. Sutherland and J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematics Education.* Berlin: Springer.

Clark, H. H. and Carlson, T.B. (1981). Content for Comprehension. In J. Long and A.D. Baddeley (Eds.), *Proceedings of the International Symposium on Attention and Performance IX.* Hillsdale: Lawrence Erlbaum.

Davis, E. J. (1978). A Model for Understanding in Mathematics. *Arithmetic Teacher 26* (September), 13-17.

Davis, R. B. and McKnight, C. (1980). The Influence of Semantic Content on Algorithmic Behavior. *Journal of Mathematical Behavior 3* (1), 39-87.

Dörfler, W. (1996). Means for Meaning. To appear in: *Proceedings of ICME 8.*

Engelkamp, J.(Ed.) (1984). *Psychologische Aspekte des Verstehens.* Berlin: Springer.

Ernest, P.(1994a). What is Social Constructivism in the Psychology of Mathematics Education? *Proceedings of the 18th Conference on Psychology of Mathematics Education, Vol. 2.* Lisboa.

Ernest, P. (1994b). Constructivism: Which Form Provides the Most Adequate Theory of Mathematics Learning? *Journal für Mathematik-Didaktik 15*, 327-342.

Fischbein, E. (1987). *Intuition in Science and Mathematics.* Dordrecht: Reidel.

Fischbein, E. (1989). Tacit Models and Mathematical Reasoning. *For the Learning of Mathematics 9* (2), 9-14.

Halmos, P. R. (1981). Does Mathematics Have Elements? *The Mathematical Intelligencer 3*, 147-153.

Herscovics, N. and J. C. Bergeron (1983). Models of Understanding. *Zentralblatt für Didaktik der Mathematik 15*, 75-83.

Hörmann, H. (1976). *Meinen und Verstehen.* Frankfurt: Suhrkamp.

Johnson, M. (1987). *The Body in the Mind: The Bodily Basis of Meaning, Imagination and Reason.* Chicago: University of Chicago Press.

Kaput, J. J. (1979). Mathematics and Learning: Roots of Epistemological Status. In: J. Lochhead and J. Clement (Eds.), *Cognitive Process Instruction. Research on Teaching Thinking Skills.* Philadelphia: The Franklin Institute Press.

Krummheuer, G. (1989). Die Veranschaulichung als "formatierte" Argumentation im Mathematikunterricht. *mathematica didactica 12*, 225-243.

Malle, G. (1988). Die Entstehung neuer Denkgegenstände – untersucht am Beispiel der negativen Zahlen. In: W. Dörfler (Ed.), *Kognitive Aspekte mathematischer Begriffsentwicklung.* Wien: Hölder-Pichler-Tempsky.

Mann, I. (1991). *Ich war behindert an Hand der Lehrer und Ärzte.* Weinheim: Beltz.

Papert, S. (1980). *Mindstorms. Children, Computers, and Powerful Ideas.* New York: Basic Books.

Presmeg, N. C. (1994). The Role of Visually Mediated Processes in Classroom Mathematics. *Zentralblatt für Didaktik der Mathematik 26*, 114-117.

Schipper, W. (1982). Stoffauswahl und Stoffanordnung im mathematischen Anfangsunterricht. *Journal für Mathematik-Didaktik 3*, 91-120.

Schreiber, A. (1983). Bemerkungen zur Rolle universeller Ideen im mathematischen Denken. *mathematica didactica 6*, 65-76.

Sheehan, P. W. (Ed.) (1972). *The Function and Nature of Imagery.* New York: Academic Press.

Simon, H. A. (1969). *The Sciences of the Artificial.* Cambridge, MA: MIT Press.

Skemp, R. R. (1976): Relational Understanding and Instrumental Understanding. *Mathematics Teaching 77*, 20-26.

vom Hofe, R. (1995). *Grundvorstellungen mathematischer Inhalte.* Heidelberg: Spektrum.

vom Hofe, R. (1996). On Concepts of the Generation of Adequate Images and Basic Ideas – Normative, Descriptive and Constructive Aspects. To appear in: *What Is Research in Mathematics Education, and What Are Its Results?* Washington, D.C.: Kluwer.

von Glasersfeld, E. (1989). Constructivism in Education. In: T. Husen and T.N. Postlethwaite (Eds.): *The International Encyclopaedia of Education. Supplementary Volume.* Oxford: Pergamon Press.