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Basic Ways of Imagining and Understanding Mathematical Concepts

1. Despite immense progress in the field of mathematics didactics there are still many educators as well as teachers who adhere to a rather narrow picture of their subject, namely consisting on the whole of abstract relations between abstract objects and some calculation. For them, intuitive, vivid, enactive or application oriented ways of doing mathematics do not belong to true mathematics, but are mere approaches. The advantage of this picture is that the contents can be identified exactly, and can easily be made accessible to presentations in textbooks, as well as to empirical research on how students handle them, or to (so called) intelligent tutorial systems.

Yet, this picture is not a suitable foundation for teaching and learning mathematics (neither for doing or applying mathematics), as in it the category of *meaning* is ignored and hence the constitution of meaning is not a matter of education. — But in every thinking or learning process the individual assigns some meaning to some notion, situation or circumstances, and teachers, in particular mathematics teachers, have to take into account these processes of assignment.

Closely connected to the difficulties in recognizing and controlling the students' learning processes is the problem of matching concepts in the realms of mathematics, epistemology and psychology (which I will call 'mathematical', 'epistemological' and 'psychological concepts' respectively). — The conception of Basic Ways of Imagining and Understanding Mathematical Concepts (BIU) offers a didactical frame for this matching problem. In German mathematics education this conception has a long tradition. Rudolf vom Hofe (1995) investigated its history and found a lot of variants in the last 200 years, most of them tackling the matching problem by designing ideal normative mathematical concepts in the epistemological mode (vom Hofe names them "basic ideas") serving as models for the students' formation of concepts in the cognitive mode (which he names "individual images").

It seemed to be natural to all those educators to found their conceptions on an analysis of subject matter and to include their rich teaching experience as an empirical background. Thus they were much closer to their students than many mathematics professors at the universities or teachers at the Gymnasiums (in former times with the top 10 % of each age-group) who taught (and often still do teach) mathematics — maybe in an elementarized, but still — in a rarely modified manner as a pure discipline. On the other hand those educators, too, did often not care for what really happens in the students' brains, and, furthermore, in spite of their good ideas their efforts had only little success.

But one must admit that only since the 1970s there has been reasonable technology for thoroughly studying classroom actions, namely video recordings. Of course, even with this technology one does still not know how cognition 'really' works. Neither the mathematical formalization of thinking processes, nor the definition of man as an information processing being similar to a computer (Simon 1969) brought about much new insight in *human* cognition. But based on the talents of video technology we learned a lot about communicative and social interaction in the classroom, in particular, how mathematical meaning is implicitly and explicitly negotiated between the participants (cf. Bishop 1985).

Due to the constructivist and connectionist roots of their theories, some cognitive scientists underestimate, ignore or deny a dominant influence of the teacher and, consequently, of the subject matter on the students' learning processes. — In fact, during painstaking examinations of video taped and transcribed micro situations middle and long term effects can easily get out of sight. If one concentrates on social and communicational characteristics of a situation, the subject matter tends to play only a minor role. And comparing students' deviating verbal and non-verbal manifestations with teachers' obvious original intentions may support severe doubts in the efficacy (or even possibility) of extraneously determined learning processes. — These tendencies are supported by the researchers' aim to overcome the old theories because of their meager success.

On the other hand careful re-analyses of classroom situations under *subject matter aspects* often lead to plausible recasts or improvements (as well as to verifications) of former interpretations based on interaction-theoretical grounds. So to me it sounds unreasonable to exclude these aspects when exploring such a situation. As I pointed out before, in German mathematics didactics, for a lot of mathematical concepts there are well known elaborated teaching routines. Whether a teacher relies on such a routine or not: From the words, diagrams etc. that she or he uses, from her or his rejection or acceptance of students' answers etc., the observer frequently can disclose the teacher's own way of imagining and understanding the mathematical concept in question. (Throughout this talk, the notion of mathematical concept includes theorems, mathematical structures, procedures etc.) — Sometimes, the teacher's own imagery and understandings seem to be inadequate, or at least the teacher evokes inadequate imagery and understandings among the students. — These problems can be tackled didactically with the help of the conception of BIU, which is meant to be a theoretical and practical frame for a normative, descriptive and constructive treatment of concept formation processes.

A radical constructivist would argue that there is no adequacy or inadequacy of imagery and understandings. Here we have reached a point of discourse where there might be no agreement. For me, *adequacy* of concepts or adequacy of imagery and understandings is a useful and important didactical category. Of course, adequacy cannot be proved like a mathematical theorem. Whether a student's concept is adequate even cannot be stated uniquely, neither in the prescriptive, nor in the descriptive mode. But there are strong hints in either mode: If a student's statements about, and actions with a mathematical concept sound plausible and seem to be successful to his own common sense as well as to experts, we would concede some adequacy (for more details cf. E.J. Davis 1978).

From a didactical point of view, it is not crucial whether teachers actually 'teach' their students or whether they only *stimulate* their students' concept formation processes. Good 'teaching' always contains stimulating the students' own acitivities.

2. By the adjective 'basic' there are expressed several essential characteristics of the conception of BIU:

(1) It includes a tendency of *epistemological homogeneousness and obligation* how mathematical concepts should be understood.
(2) *Psychologically* speaking, it indicates that students' individual concepts normally are, and in the teaching processes the epistemological concepts should be, *anchored in* the students' *worlds of experience*.
(3) With respect to *subject matter* it stresses the importance of *fundamental ideas* (in the sense of Halmos's elements, 1981, or Schreiber's universal ideas, 1983) guiding the study of any mathematical discipline.

Epistemological homogeneousness: This tendency seems to be in contradiction with modern pedagogical and didactical paradigms like "the students should create their own mathematics", or "the students have to find their individual ways in solving mathematical problems" etc. In fact, teaching does not mean telling (Campbell & Dawson 1995), but it means stimulating students' cognitive activities, negotiating mathematical meaning in the classroom etc.

But this way of conceiving the teaching-learning process does not entail any obligation for the teacher to tolerate or even to support inadequate individual concepts, on the contrary: it makes the teacher's task much more difficult. She or he must be provided with a good theoretical and practical competency in mathematics, mathematics applications, epistemology, pedagogy, psychology, social sciences etc., in order to

- develop her or his own view of the epistemological kernel (which must not be identified with a mathematical definition) of some mathematical concept which the students shall acquire,
- perceive the students' actual individual concepts as truly as possible and to judge their adequacy,
- help the students, if necessary, to improve or to correct their individual concepts into adequate ones near the epistemological kernel,
- possibly learn by the students and improve her or his own individual concepts.

This task imparts a predominant role in the teaching-learning process to the teacher's own imagery and understandings and to their transposition into didactical action. For example, if for the calculation of the number π a circle is approximated by a sequence of polygons and the teacher uses a phrase like "in this sequence the polygons have more and more vertices, and finally they turn into the circle", the students' formation of an adequate concept of limit is obstructed.

The epistemological kernel of a concept corresponds to a commonly shared socio-psychological kernel. Such a socially constituted kernel is an important prerequisite for the construction of individual argumentation and its introduction again into classroom interaction (cf. Krummheuer 1989). — It is obvious that this commonly shared kernel should be as extensive as possible, which, again, gives the teacher a central position in the teaching-learning process.

Anchoring mathematical concepts in the students' worlds of experience: Even working mathematicians need some real world frame for doing mathematics ("we consider ...", "if x runs through the real line ..." etc.; cf. Kaput, 1979, and many others). All the more students need such frames so that they can constitute meaning with the subject matter they are about to learn (Davis & McKnight, 1980, Johnson, 1987, Fischbein, 1987, 1989, etc.). As such frames do not belong to the epistemological concepts, the teacher is rather free when constructing real world situations where basic imagery and understandings can be unfolded.

These situations need not be absolutely realistic, on the contrary, by alienating them with the help of fairy-tale traits and concentrating on the essence they can be turned into metaphors with their explanatory power. One can take human beings, animals, things, which are more or less anthropomorphized and more or less mathematized. These participants of the situation have to act somehow, following some arbitrary rules, pursuing some arbitrary plans, obeying arbitrarily physical and other natural laws, or not.

Let us take as an example the *arithmetical mean* $\overline{a}$ of n (positive) numbers a_1, a_2, ..., a_n, defined as $\frac{a_1+a_2+...+a_n}{n}$ and characterizing this collection of numbers. The situation can be visualized on the number line, where the numbers are spread over a certain section. One looks for the center of this section, taking into account the distribution of the numbers.

This visualization does not really fit the definition. In fact, the definition is connected to some other imagery: We think of n people with the n numbers as their respective body-heights. We build a tower by putting the n persons one on another. Then we look for one more person whose height $\overline{a}$ is given by the following conditions: If we build a second tower with n copies of this person, it has to have exactly the same height as the first tower. This leads to the equation $n\cdot\overline{a} = a_1 + a_2 + ... + a_n$, which can easily be transformed into the above definition by dividing both sides by n. In our visualization this means dissecting the first tower into n equal parts. — Of course, one needs not such drastic illustrations; arrows on the number line do as well.

But there seems to remain some missing understanding: By adding those n numbers one gets far away from that section where these numbers are located. — Are we sure that by dividing the sum by n we return to that section and arrive at a number which can reasonably be called mean value of the original numbers? The same question arises, when we transform the equation into $\overline{a} = \frac{1}{n}\cdot a_1 + \frac{1}{n}\cdot a_2 + ... + \frac{1}{n}\cdot a_n$, where $\overline{a}$ is the weighed sum of the a_k: Each of the n persons gives $\frac{1}{n}$ of her or his height, and the average person is constructed with these n parts. With these small numbers, again, one leaves the original section, now in the direction of 0, and, again, it is questionable whether summing up these numbers brings us back into its centre.

Let us take into consideration another fundamental attribute of the arithmetical mean, i.e. its compensatory character: Some of the numbers are larger, others are smaller than their arithmetical mean, to be more exact: The arithmetical mean is determinated by the condition that the sum of the positive deviations is just compensated by the sum of the negative deviations, this determination being appropriately visualized on the cover of the journal "mathematik lehren" (fig. 1). — For the equation this means to subtract on both sides n times $\overline{a}$ and to assign to each number a_k its own copy of $\overline{a}$. We get $0 = (a_1 - \overline{a}) + (a_2 - \overline{a}) + ... + (a_n - \overline{a})$ with the disadvantage that there are involved neagtive numbers: The sum of all (oriented) differences is 0. In order to avoid negative numbers one can put the given numbers in their natural order and one finds an index number m, for which the following holds: For $k \leq m$ we have $a_k \leq \overline{a}$, and for $k > m$ we have $a_k > \overline{a}$. —

And now we can avoid negative numbers: $(\overline{a} - a_1) + (\overline{a} - a_2) + \ldots + (\overline{a} - a_m) = (a_{m+1} - \overline{a}) + (a_{m+2} - \overline{a}) + \ldots + (a_n - \overline{a})$.

This version of the equation is still unsatisfactory as it uses the value of the arithmetical mean which is usually not known from the beginning but shall be calculated. This problem can be tackled with the help of an algorithm, which was prepared for the primary school by my colleague Hartmut Spiegel (1985). From now on I assume that the numbers a_k are whole numbers; but the algorithm can easily be generalized. — Reduce the largest number by 1 , and, at the same time, increase the smallest number by 1 . This double action results in a new collection of numbers which differs from the old one by two entries and which has the same arithmetical mean. It has to be repeated with this new collection, and so on. (If there is more than one maximal or minimal number, one chooses one out of them for the reduction respectively for the enlargement.) With each step the sums of the positive and of the absolute negative deviations from the (not yet known) arithmetical mean are decreased, and at the same time it is clear that the arithmetical mean lies between the smallest and the largest number (fig. 2a).

If one wants to work with this algorithm in the primary class room, body-heights are not appropriate as they belong to a continuous number domain. Rather one has to choose domains with natural units like the heights of Lego block towers, amounts of money, or numbers of mistakes in a test. Spiegel's central example had the advantage of representing a piece of meaningful applied mathematics: How many matches are there in a box in the average? In this context the algorithm means at each step: to order the boxes according to how many matches they contain and then to take one match out of the fullest box and put it into the emptiest one.

The algorithm ends, when all boxes contain the same number of matches. Then the arithmetical mean is found, as it has the following meaning, not only in the primary school: If we would collect all matches from all boxes and then distribute them *evenly* on all boxes, how many matches would be in each box? — Of course, it can also happen that the algorithm reaches a stage, where there remain *two* kinds of boxes, one kind with a maximal number of matches and one kind with a minimal number of matches, the maximum and the minimum differing just by one match. If we would then continue the algorithm, nothing would change, as transferring one match from a 'maximal' box to a 'minimal' one would only turn this maximal box into a minimal one and vice versa. — So this is a second kind of final state. Here the arithmetical mean is not a whole number, but lies between the two numbers which are still involved. From the ratio of the two frequencies one can tell whether the arithmetical mean lies nearer to the one or to the other number. — As long as there are numbers (of matches in boxes) which differ by more than 1 , the algorithm can and must be continued.

Spiegel's example which he realized with ten years old students had roughly the following diagramme (fig. 2b). If one starts working immediately in the iconic mode of representation, one tends to move one complete box from the right column to the left column. This is a severe mistake, as there are mingled two object levels and two kinds of frequencies. One treats each column as one box and its height as the number of matches in it. But, in fact, there are 40 boxes which are ordered according to the number of matches they contain, and each column consists of all boxes with the same number of matches. — This mistake is one of the most favourite ones to be made in statistics, and it regularly blocks the learning of this discipline from the beginning.

Of course, the activities have to start (and to remain for a long time) in the enactive mode: The boxes have to be ordered and to be put together to (horizontal) columns *actually*. Then one box on the right column and one on the left column have to be opened, one match has to be removed from that right box and to be put into the left one, and then the two boxes have to be moved to their respectively next columns.

This example also shows the problems of genuine mathematics application in school: The algorithm is very time expensive (in this small example it needs 32 steps), and it puts a great strain on the students' perseverance.

So far, I presented four or five ways of imagining and understanding the concept of arithmetical mean. I think, the students should make them all their own in the course of their school time, and they should relate them to each other and to other concepts (e.g. the mean deviation). — It should become clear on the one hand that even a seemingly static concept like the arithmetical mean is basically grounded on actions, and on the other hand that *symbolic* and formal mathematics also plays a constituent part in relating different ways of imagining and understanding a mathematical concept to each other.

Fundamental ideas for mathematical disciplines (in an epistemological and psychological sense): Basic imagery and understandings are not only meant as a peg on which to hang some mathematical content, but they shall lay the foundations for further meaningful interpretations of concepts within a mathematical discipline.

3. The notions of imagery and understanding stand for two fundamental psychological constructs. There exists an extensive literature about them. Different authors have different definitions, most of them not very concise. A lot of contemporary cognitive scientists disregard these two constructs anyway, as they escape hard empirical research and do not fit a computer related view of intelligence. — But it is just these — seen behaviouristically — shortcomings, their vagueness and flexibility, which turn these constructs into suitable means for analyzing (and promoting) such complex didactical objects like human teaching-learning processes.

Imagery can be grasped as: mental, often visual (but also auditory, olfactory, tactile, gustatory, and kinesthetic; cf. Sheehan 1972) representations of some object, situation, action etc. having their sensory foundations in the long term memory and being activated in *conscious* processes. A person activating some imagery has already some meaning, some intentions in mind and organizes these processes according to these intentions (Bosshardt 1981). — Imagery is closely related to intuitions, but its objects are more concrete, and meaning plays a more important role.

The objects of imagery (and understandings) can be given in different modes, namely analogous or propositional. I don't want to resume the cognitive scientists' quarrel in the 1970s about the interrelations between these two modes or about their separate existence as ways of thinking. In my opinion both are valuable means for analyzing imagery and understandings in teaching-learning processes.

Apparently, imagery is more closely connected to the analogous mode, and understandings are more closely connected to the propositional mode of thinking. But it is difficult for a person to activate some imagery without propositional elements, in particular in didactical situations, as in these situations verbalization is *the* fundamental means for a participant to communicate either with others or with her- or himself (this communication with oneself being a transposition of a social situation to one's mind which is typical for teaching-learning processes). On the other hand, there can be no process of understanding without recurring to any plausible imagery and to analogous elements.

Obviously, thinking in the analogous mode can be stimulated by analogous means like pictures, diagrams etc. (with a lot of limitations; cf. Presmeg 1994), and the propositional mode can rather be stimulated by propositional means like verbal communication. In the age of paper and pencil and of books, analogously given objects frequently are of a visual, static nature, and the learners have to undertake some effort to make these 'objects' plausible, meaningful, vivid imagery matching their worlds of experience. In the nearest future the use of multi-media in schools (in the western world) possibly will relieve the students from these efforts.

Whether multi-media will be conducive to the students' learning processes, is not yet settled: The students' inclinations and abilities to undertake efforts to generate mathematical concepts could be undermined. — This problem is complementary to the following classical one, related to the use of visualizations (diagrams, icons etc.): Among many educators there exists the naive belief that visualizations do facilitate the students' learning processes automatically. But as, for example, Schipper (1982) showed with primary graders, many visualizations are not self-explanatory at all, but they are subject matter which has to be acquired for its own sake on the one hand, and in relation to the visualized contents on the other hand. — As a matter of course, visualizations can be successful didactical means, but not because they would reduce necessary effort, but because they demand *more* effort and give hints how to direct and structure this surplus effort and thus make it effective.

There are didactical situations, as well as mathematical concepts, as well as students, for which resp. for whom one of both modes is more suitable. For teaching and learning mathematics it is important that there has to be a permanent transformation between the two modes. Maybe geometry can be treated predominantly in the analogous mode, and algebra in the propositional mode, maybe the teacher is even able to take into consideration the preferences of single students. But on principle, both modes must be present.

Taking into account the wide spread propositional appearance of mathematics teaching, in particular on the secondary school level, there is need of an increased use of the analogous mode all over the world. — By stressing the students' anchoring of their individual concepts in their worlds of experience, the conception of BIU lays some accent on the analogous mode, as a prophylactic counterweight to the preponderance of the propositional mode in the upper mathematical curriculum.

The psychological construct of *understanding* is still more complicated and non-uniform. For didactical reasons the following aspects are relevant:

(1) One can understand people, their actions, situations, the motives or the aims of the participants (practical knowledge of human nature, *common sense*).

(2) One can understand utterances *medially* and *formally* (e.g., if they are made loud enough and in a language one knows).

(3) One can understand the *content* of a *message* made by someone (understand what this someone *means* by a certain communication, text, phrase, word, symbol, drawing etc.).

(4) One can understand technical matters, working principles of gadgets, mathematical structures, procedures etc. (*expertise*).

At first glance, aspect (4) seems to be most suitable for the conception of BIU. But it becomes immediately clear that each of these aspects is important for the learning of mathematics and has to play an essential role in the conception, in particular (3). This aspect is a classical psychological paradigm, but the general opinion about it has changed: Today, one does not believe anymore that it consists just of finding some objective meaning of given signs, but that the receiver of a message tends to and has to embed the message in some context and, in doing so, tries to reconstruct its meaning (cf. Engelkamp 1984), thus getting near aspect (1).

It goes without saying that there is no understanding (3) without (2): the sender and the receiver of a message have to have a common language, not only in a direct, but also in a figurative sense: As Clark & Carlson (1981) put it, there has to be a "common ground", which, again, refers to aspect (1). — In school teaching, and in particular in mathematics teaching, the common ground of teachers and students is often rather thin, if existing at all. — But, extending the common ground does not only mean that the students have to be better instructed so that they make the teachers ground their own. Rather the teacher must engage in the students, attach importance to them (and not only to the subject matter), understand them as human beings (again, aspect (1)), and try to reconstruct or to anticipate their ways of thinking.

By following the conception of BIU, to some extent the teacher is forced to do so, and furthermore, her or his expertise can be promoted. But this way of teaching and learning demands much more effort for both parts, in comparison with the usual way, where teachers, in good harmony with the students, are satisfied with students' instrumental understanding (in the sense of the late Richard Skemp 1976).

In the following example, the teacher (resp. the researcher) did not quite understand the student's way of thinking. It was originally described by Malle (1988) and re-analyzed by vom Hofe (1996): In order to develop the concept of negative numbers, Ingo, the student, was given the following situation: "In the evening the temperature is 5 degrees (Celsius) below zero. During the night a warm wind moves inland, and the temperature rises *by* 12 degrees. — What is the temperature next morning?" Ingo answers correctly: "7 degrees", but in the dialogue with the interviewer, he shows inadequate imagery. When he shall sketch the situation, he asks whether he must draw three thermometers, and later he explains that at midnight the temperature went up to +12 degrees, and in the morning it dropped to +7 degrees (fig. 3).

Malle gives well known and, of course, correct explanations for Ingo's obvious inadequate dealing with the situation: Ingo is not able to identify the elements which are important for solving the problem, but invents additional information and tells fairy tales, and he does not differ between the starting and the final state (i.e. the starting and final temperature, represented on the thermometer) on the one hand and the change between the states (the rise of the temperature) on the other hand.

In his careful re-analysis, vom Hofe shows that the problem lies in Ingo's way of imagining and understanding the physical situation, which is no suitable basis for the formation of the mathematical concept. Whereas the interviewer expects Ingo to focus on the changes of the mercury column (as a direct model of the number line), Ingo imagines two masses of air, a cold and a warm one, which mix and result in a third mass with average temperature. Therefore he needs *three* thermometers, and in the night the temperature does not rise *by* 12 degrees, but *up to* 12 degrees, and goes down again in the morning. The idea of mixing air masses is, physically speaking, not at all inept, but it merely does not fit the mathematics that the interviewer has in his mind. For Ingo, there are two states of temperature which result in a third one, the weighed arithmetical mean, and not one state which changes into another.

Granted that every human being tends continually to conceive, or to make and to keep her or his environment meaningful and sensible, one must admit that usual mathematics teaching in large parts has a contra-productive effect. — The strive for "constance of meaning" (Hörmann, 1976) is in my opinion a characteristic trait of humans, which, for example, is largely ignored in Piaget's biologistic theory of equilibration.

Mathematics teaching, too, is such an environment which humans who are in touch with it try to make meaningful and sensible for themselves. As an *extreme* example, (in a famous French movie from 1984) in a physics lesson in Paris the absent-minded student from Algeria understands "le thé au harème d'Archimède", when he hears "le théorème d'Archimède" (which means: "tea time in Archimedes's harem" instead of "the theorem of Archimedes"). Even if we omit such extreme cases, it still seems to be rather normal all over the world that students tend to develop their own non-conformistic imagery and understandings, which, however, often remain implicit.

Fischbein (1989) calls them "tacit models" and characterizes them as simple, concrete, practical, behavioural, robust, autonomous and narrowing. Their robustness results from their simplicity, their anchoring in the students' worlds of experience, and their short term success with convenient applications (see the example of Ingo). Inadequate tacit models come into being because of lack of adequate basic imagery and understandings, which in their turn would also be concrete, practical, etc., successful and therefore robust, and *not* narrowing, but capable of expansion.

So the conception of BIU includes the strategy of occupying the students' frames with adequate basic imagery and understandings from the beginning, i.e. to give them the possibility and to enable them to develop such imagery and understandings by themselves.

Nevertheless, students will still generate a lot of inadequate tacit models, and teachers must be able to recognize them and to help the students to settle them. In this, again, the teachers can be supported by the theoretical and practical frame of the conception of BIU, thus using the constructive aspect of the conception (as vom Hofe, 1995, puts it).

Fischbein (1989), like many other educators and cognitive scientists, recommends that the students should undertake meta-cognitive analyses in order to discover and eliminate the defects in their frames. — I couldn't find evidence in the literature that students would be able to successfully analyze their own (wrong) thinking without massive interventions by the teacher or by some interviewer. According to my own experience with young people in all grades, they are overstrained if they shall reflect reflexively about their own reflections.

In fact, in many classroom situations there can be found actions of understanding on a meta-level; for example, if students recall how they solved a certain problem, or if they try to find out the teacher's intentions, instead of trying to understand the contents of her or his statements. But, in general, this kind of understanding (aspect (1)) is not explicitly reflected by the students.

One essential trait of every didactical situation is (or should be) that the participants strive for understanding the contents of some message given, verbally or non-verbally, by the teacher, students, the textbook etc. (aspect (3)), with the underlying aim that the students shall acquire expertise (aspect (4)). Whereas aspect (3) stresses the *processes* of understanding, aspect (4) stands for the *products* of these processes. The products are not only results, but at the same time they are starting points for new processes, and each understanding process starts on the ground of some already existing understanding.

In mathematics teaching, both aspects of understanding ((3) and (4)) deal with the same objects: the messages, seen ideally, deal with mathematical concepts, about which the students shall acquire expertise. — In the humanities and in the social sciences, as well as (in an indirect way) in mathematics, this expertise again often refers to social situations (in a wide sense) and thus is in parts identical with aspect (1). — So, finally, in normal teaching-learning processes all the aspects of understanding discussed here belong together and are essential for success.

In my view, there is no understanding without imagery, and no imagery without understanding. With the notion of imagery there are stressed the analogous mode, roots in everyday lives, intuitions etc., — whereas with the notion of understanding there is laid some accent on the propositional mode, on subject matter, on predicates etc., — but both notions do not only appear together, rather they have a large domain of essence in common.

4. There can be identified roughly four types of BIU for the use in mathematics teaching in the primary and secondary grades:

(1) More or less *global* BIU, especially for the formation of the concept of number and for elementary arithmetic: multiplication as repeated addition; division as partitioning (splitting up; 'Aufteilen') or distributing (sharing out; 'Verteilen'); fractions as quantities or as operators, negative numbers as states or as operators, the machine model for operators, the little-people metaphor for running through an algorithm. Basic imagery and understandings are not bound to primitive, non-quantifiable actions (in the sense of intuitive understanding according to Herscovics & Bergeron, 1983), and their formation is not a kind of mathematical propaedeutics or pre-mathematics, but — in my opinion — genuine mathematics (just without calculus with symbols). They would be useful in the upper secondary grades as well, for example with the concept of limit and infinitesimal thinking as a whole.

(2) More or less *local* BIU, e.g. the arithmetical mean, the internal rate of return of an investment, the circumcircle of a regular polygon.

(3) BIU for *extra-mathematical* concepts, situations, procedures (from physics, economics, everyday life etc.), which are to be used in mathematics teaching (example: Ingo and the temperature).

(4) BIU for *conventions*, e.g. the meaning of symbols, or of diagrams. Example: The teacher tries to explain subtraction with the help of the following situation: "Mother baked six cakes for her daughter's birthday; the dog Schnucki ate four of them. How many are there left?" She draws 6 circles on the blackboard and crosses four of them out (each with one line), hoping to support visually the understanding of the problem 6–4=2 (fig. 4): But Ralph, a learning disabled child, wonders why the teacher cuts the marbles into halves (Mann 1991).

It goes without saying that the prototypes, metaphors, metonymies (Presmeg 1994) used for BIU should not obscure the concepts they refer to, or even falsify them, as it is the case with the concept of circle in Papert's (1980) original idea of *turtle geo-*

metry. The children shall draw a circle by programming the turtle to draw a straight line of length 1 , then to do a right turn of the amount 1 , and to repeat these two actions 360 times , i.e. by drawing a regular polygon with 360 vertices. In fact, the result looks like a circle, but the way in which it was produced belongs to a concept which is essentially different from the Euclidean circle. It's true, that every line on the computer screen is a sequence of squares; but this is not the point, as students with some experience with paper and pencil as well as with computer screens will recognize the shortcomings of any realization of geometric forms and will be able to idealize these forms, if at least the underlying activities are appropriate. — But the procedure for making a Logo circle is not appropriate for Euclidean geometry. — Furthermore, I doubt that the Logo geometry is a good preparation for differential geometry, — eventually it is a helpful model for someone who already has the concept of mathematical limit at her or his disposal, whereas it is likely to be a mental obstacle for someone who is still on her or his way of acquiring this concept, let alone for primary graders.

Basically, the *area* of a polygon is the answer to the question: How often does the unit square fit into the polygon? The polygon has to be *measured* by the unit square. — But even if the teacher stresses such measuring activities, students often form inadequate BIU of the concept of area, when and because the teaching focusses on the formula $A=p\cdot q$ for the rectangle with side-lengths p and q . Teachers then tend to teach the area as multiplication of lengths, which is a correct, but rather deep understanding of the concept, which should be grasped by the students later, but which is not accessible to them at the beginning.

First of all, the formula just expresses the multiplication of two pure numbers, namely the both numbers of unit squares fitting into the two sides of the rectangle respectively. — In the classical German curriculum there is paid much attention to this interpretation of the formula, but in my opinion the concept of area is obscured by an over-methodization (fig. 5): The rectangle is tiled with unit squares which are grouped into p stripes each containing q squares, thus suggesting to the 10-years-old students that they have to apply the operation of multiplication. Of course, these BIU of multiplication are correct and adequate for — multiplication, still they are by no means new to these students, but were acquired by them two years before, when they had learned the operation of multiplication and worked a lot with rectangular fields. The seemingly new treatment of multiplication now (without telling that it is a mere repetition) suggests to the students that there must be something special in connection with the concept of area. But this 'something special' remains a mystery to them, because — as *we* know — there *is* nothing special. Using multiplication is just an economic way of counting.

In my own lessons I show the students a sketch of two differently shaped garden plots whose sizes they shall compare by tiling the plots with unit squares and counting the squares. I use to embed the problem in a story about two children who have to mow their lawns and want to know who has to work more (of course, this does not only depend on the area). After this introduction I give the students some work sheets with which they can make the essential discoveries themselves. The only hint is that they shall count in a *clever* way. Most of the students are able to work out the basic ideas on their own: Divide the garden plots into rectangles, tile them *virtually* with 'unit' squares, calculate their areas by trivial multiplications and additions, and finally compare them (fig. 6). Later they also see that for this comparison one has to use squares of the *same* size.

Transformation geometry: When in the late 1960s and early 1970s transformation geometry was pushed into the mathematics curriculum, it was assumed that real motions of real objects could serve as BIU. In fact, the students accepted these BIU willingly and transferred them easily into continuous motions of point sets in the plane. But the crucial point was the abstraction of the motions, which the students in general did not manage to perform. Their BIU of transformations grounded on motions were so robust that the good advice to focus their attention to the starting and final positions of the geometric forms or to the plane as a whole remained useless, because, for example, the notions of starting and final position, again, evoked imagery about motions (cf. Bender 1982).

Thus, mathematicians and mathematics educators failed to establish in the curriculum the full algebraization of geometry by transformation groups, and up to today geometric transformations are not treated as objects on their own, but only as means to investigate geometric forms. The idea of embodying Piaget's groupings of thinking schemes in geometric transformation groups had proved to be too naive.

There are reasonable applications of continuous motions, e.g. in good old *congruence geometry* by Euclid and Hilbert: Two geometric forms are congruent, if they can be moved so that they exactly cover one another. In German there is a synonym for the word 'kongruent' which is due to this reciprocal covering (='decken'), namely 'deckungsgleich' ('gleich'= 'equal'). In congruence geometry, different from transformation geometry, the specific form of these motions is not essential at all. So the students need not, cannot and, in fact, do not memorize them, and motions are not likely to turn into mental obstacles against viewing congruence as an interrelation between two stationary geometrical forms.

For *functional reasoning* in geometry and other mathematical disciplines, like calculus, there is needed a different, and slightly more abstract, concept of motion: What happens in the range of a function, if one 'walks' around in its domain? Example: The area function assigns to each triangle of the Euclidean plane its area. Starting with one triangle, one changes one of its vertices, and one observes, how the area changes. — The metaphorical character of this situation is obvious: There is a space (the domain, i.e. the set of all triangles) and someone or something (the variables) who 'walks' around; and the motions of this someone or something are transferred by some mechanism, like an abstract pantograph, into another space (the range, i.e. the positive real numbers).

Here we have a good field of application for modern so called '*Dynamical Geometry Software*' (DGS) like Cabri Geometre: The transformation mechanism is hidden in the programme, and the connection between the changes in the domain of a function and the resulting changes in its range are visualized in a direct and lucid way. These possibilities, of course, refer also to transformation geometry with the Euclidean plane as domain and range. In my opinion, it is the first time that this subject could be taught to 13-years-old students in a not misleading way, namely without continuous motions.

Unfortunately, one cannot eat the cake and keep it at the same time: Observing these connections on the computer screen seems to lessen the students' motivation for further investigations. What is more, the supporters of DGS have higher aims in their mind, as they think, quite rightly, that the time expensive use of computers cannot be confined to BIU. That is why, as far as I know, there is still no modern lower secondary school curriculum for transformation geometry on the grounds of DGS.

5. In all times, all over the world, mathematics educators reflected and still do reflect on basic ways of imagining and understanding mathematical concepts, though they usually do not name them like that and possibly have different or no conceptual frames. There is still missing a theory unifying the relevant disciplines 'mathematics', 'epistemology' and 'psychology'. The work of vom Hofe and my work is one attempt.

But the realization in didactical and teaching practice is at least as important as the theory. Which basic imagery and understandings do we think to be adequate? How can we support the students generating adequate basic imagery and understandings? Which inadequate imagery and understandings can occur? How are they caused? How can they be improved or corrected? — In my opinion, these are fundamental questions of mathematics education.

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Mittelwerte

Fig. 1

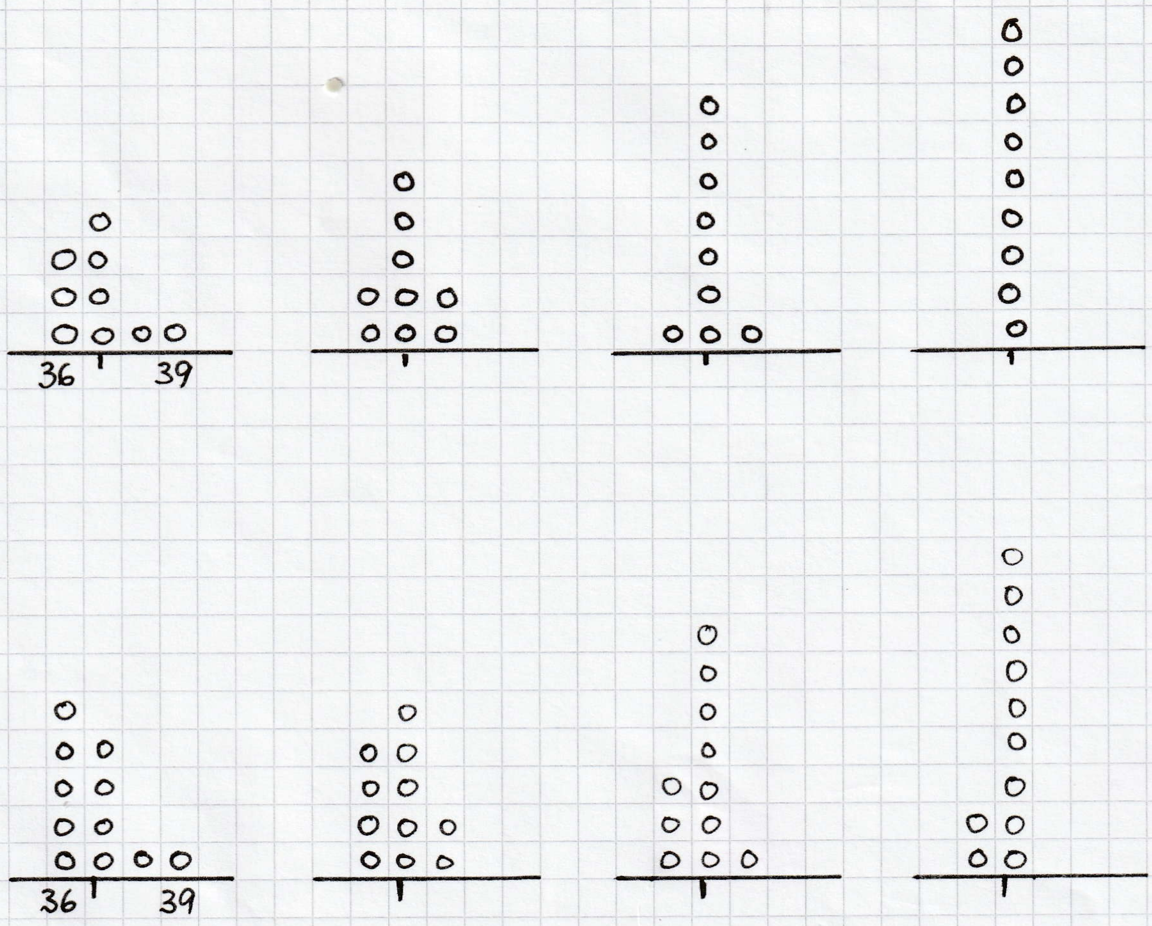


Fig. 2a

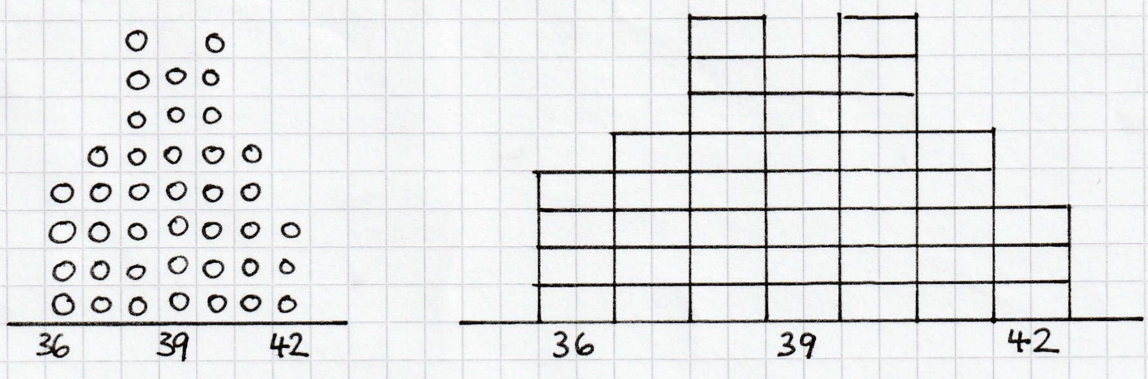


Fig. 2b

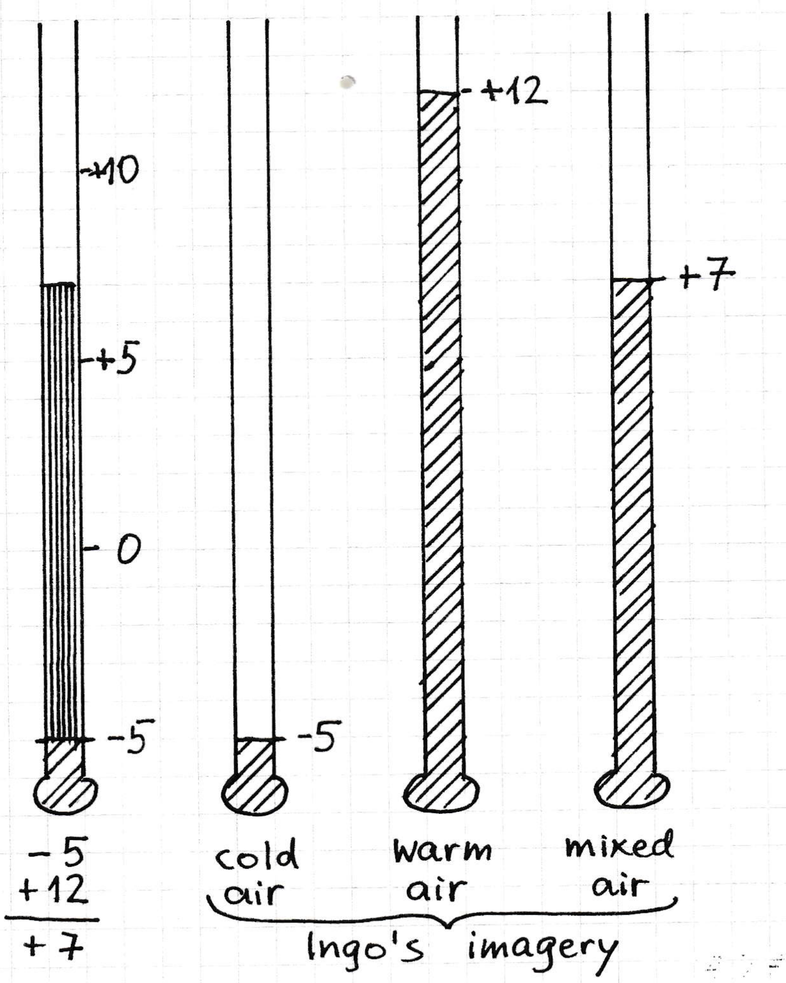


Fig. 3

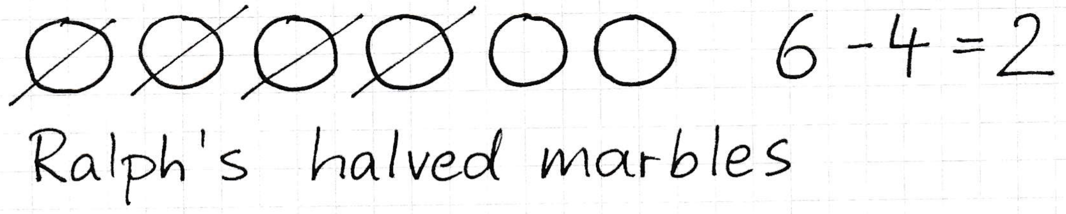


Fig. 4

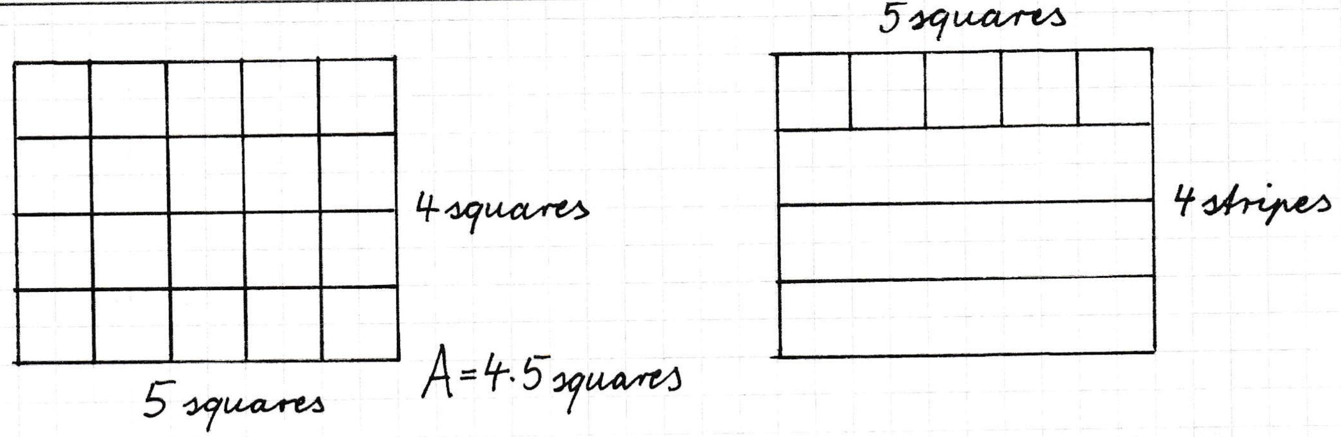
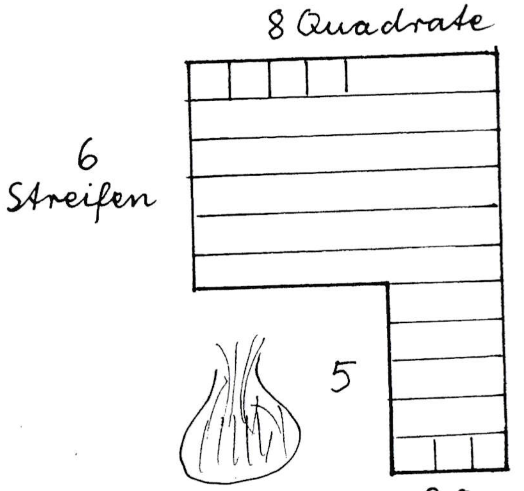


Fig. 5

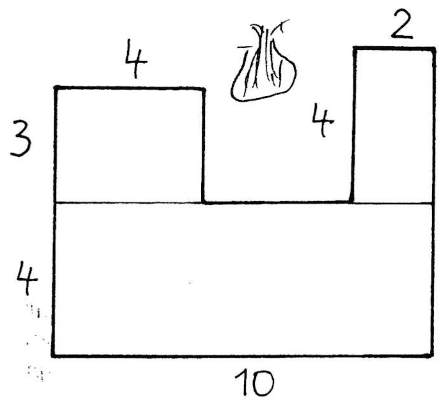
Fig. 6

Sarah



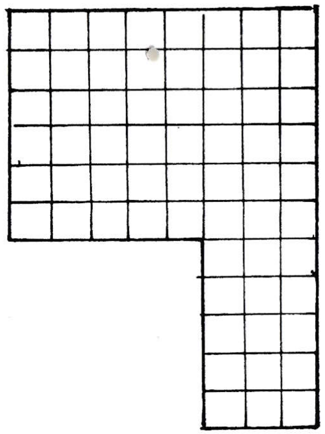
$6 \cdot 8Q = 48Q$
$5 \cdot 3Q = 15Q$
$63Q$

Patrick

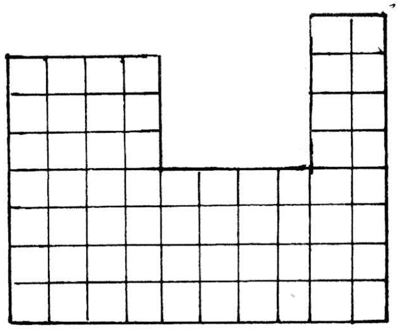


$4 \cdot 10Q = 40Q$
$3 \cdot 4Q = 12Q$
$4 \cdot 2Q = 8Q$
$60Q$

Sarah



Patrick



Yannick