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Some paradigms of mathematics education — and how work with computers is related to them (with particular consideration to the teaching of geometry)

Abstract:

Mathematics didactics are influenced by many other disciplines. Adopting their results and methods without adapting them to the concerns of mathematics teaching often entails only short term progress, if any at all. In this paper several questionable paradigms originating from other disciplines are discussed together with how they are modified or reinforced by the inclusion of work with the computer. These considerations are put into practical terms in the case of teaching geometry using so-called Dynamic Geometry Software (DGS). Careful investigations of the positive and negative effects are still needed.

The paper is based on lectures given by the author at the 8th International Symposium on Mathematics Education at Klagenfurt, Austria, (see Kadunz et al. 1998) and at the annual meeting of the GDM 1999 at Bern, Switzerland, and mainly reflects the situation in German mathematics didactics. Thanks to John Searl in Edinburgh and Neveille Neil for their valuable comments and their help with the translation.

Mathematics didactics are influenced by many other disciplines (pedagogy, general didactics, philosophy, psychology, social sciences, mathematics, etc.) and have to take into account, and even incorporate, their results and methods. At the same time however it is a scientific discipline of its own with its own methods and results, comprising (at least) two different traits, namely descriptive and normative, both directed towards theory and practice at the same time. Didactical research includes not only empirical, but also 'engineering' methods (as Wittmann, 1995, puts it). Of course, no mathematics didactician can cover the whole field, but they should take notice of the work of others within the field, possibly based on different scientific paradigms, e.g. the experience of mathematics teachers (Wittmann 2000), and not reject others' ideas from the beginning. In particular they should not only accept those outcomes from research which fit their own theories (assumptions, beliefs), but also those which do not support them or which even contradict them. On the other hand, if methodologies from other disciplines are used puristically, i.e. without adapting them to mathematics didactics, for example ignoring the mathematical, epistemological and/or cognitive structure of the involved specific subject matter, they may well result in surprising short-term findings. However, in the long run, they will be of little use for really improving **mathematics** teaching. Such extraneous paradigms are, for instance:

(i) Taking statistical methods from medical, psychological, economic or social research fields and often applying them to (the outcomes of) teaching-learning processes without controlling, let alone establishing, such fundamental considerations as

- how representative the sample is
- the independence of the investigated attributes, and in particular
- the validity of the questions.

(ii) Imposing a formalistic mathematical-logical structure on the collection of objects which are involved in the teaching and learning of mathematics (subject matter, cognitive operations, social actions etc.).

(iii) Reduction of human beings to information-processing beings, thus forcing a close relationship between artificial and natural intelligence.

(iv) Neglecting social and other non-cognitive effects on mental processes.

(v) Transporting philosophical and political theories and ideologies (for instance 'constructivism' or 'situated' or 'social' or 'collective learning') into the field of education, and claiming pedagogical and didactical conclusions which are based on a minimal body of experience, if any at all (see Anderson, Reder & Simon 1995, 1996, 1997). (Of course, such descriptions of learning and understanding can provide insight in the classroom situation and in learning processes, but they are only partial understandings.)

(vi) In close connection with (v): over-estimation of students' and under-estimation of teachers' contributions to the learning processes.

(vii) Again in close connection with (v): disregard for the subject matter as a major influencing variable in the teaching and learning of that very subject matter.

These shortcomings have been the daily bread in the mathematics didactics community since the 1960s, and the quarrel about the so-called New Math was one of the first significant examples. It goes without saying that corresponding deficiencies adhere to other scientific disciplines as well, perhaps with the exception of mathematics.

With the promotion of the computer as a tutor, a 'tutee', or as a means or a medium for mathematics teaching (and as a device for the science of mathematics teaching), the psycho-socio-scientific structure of mathematics didactics has been given a fresh veneer, but has not essentially changed. One can go through all the items listed above, and one will find a computer-based variant or reinforcement. I will indicate just a few:

(ii) with (iii) and (iv). There is nothing wrong in modelling human intelligence or the human brain and neuronal system with a computer language (i.e. mathematically, ultimately). The fundamental fallacy consists of confusing the computer model with human cognition (as an abstract concept!). This identification may be a paradigm of the artificial intelligence movement, but I cannot find evidence that it is of use in the education of children and adolescents in particular, if one strives to avoid (iv).

(v) with (vi) and (vii). Surprisingly, parts of the computer-into-education camp (CIEC) claim to benefit from those 'soft' pedagogical paradigms mentioned above, based on simple lines of reasoning like:

(a) From a constructivist point of view, the person who is originally called 'teacher' has to confine herself or himself to the function of an organizer and moderator. Hence the students take responsibility for their own work, and the computer is the ideal gadget to make the students independent from the teacher's direction.

This argument misses several hard facts. As the teacher is replaced by the computer, the students' actions are still determined from outside and not only by an electronic appliance with all its well known technical, psychological and social shortcomings, but also by programmers who, when programming either Logo microworlds (in Papert's, 1980, language), computer algebra systems or 'realistic' scenarios, did not have in mind that special student and that special situation in which their computer programs are used. In the end students' actions are still determined by society, the school administration and the teachers as well, who will stay in control of what the students are offered even if schools were abolished and learning was to happen at home.

The willingness and ability of young people to take it upon themselves to learn something independently is very restricted, as they lack experience, knowledge (as a requirement for acquiring new knowledge) and, last but not least, insight in the necessity of learning this particular thing or of learning anything at all. Whether human teachers in general deal well with this basic pedagogical challenge can be doubted, but that computers do not has been appreciated even by the inner circle of the Logo community (see e.g. Hoyles & Sutherland 1990).

Mankind has not only the longest adolescence time biologically among all creatures, but in our complex modern societies there is also need for an intensive social, intellectual, emotional etc. maturation of every individual. For the overwhelming majority, education by parents alone would be grossly insufficient (and impossible). Schools are an important part of the educational system and, as such, part of the real world. On the other hand, they themselves (as well as the knowledge, ways of thinking, attitudes etc. which they impart) are only models of the real world. Clearly, even if these models are poor, they would be immensely more impoverished if a leading role in education were assigned to the computer (in its actual application, even including virtual reality etc.).

(b) When students are to work with computers in the classroom (which happens rather rarely in Germany but more frequently in Austria), quite often two or more students have to share one keyboard and one screen because there are less computers than students and/or there is a lack of space. This necessity is regularly claimed to be a virtue since it allegedly promotes 'social learning', 'key qualifications' (communication, working in a team etc.), responsibility for others, better learning, etc.

Even if one agrees to these rather vague ideas, there is no evidence of great success. The actual work often concentrates on a small subset of the students. If better students 'help' the weaker ones, they often tend to execute the task completely, thus preventing the weaker ones from grasping the subject matter in question (cf. Vollmer 1997). When trade and industry urge the schools to promote teamwork, they have in mind above all that school-leavers should be able to fit into a team, and this is quite the opposite of how that qualification is understood in the pedagogical paradise. Working collectively can result in a loss of concentration and this drawback can be intensified by the computer screen with its bullying effect (i.e. a permanent call for action). Thus the learning of some subject matter, in particular of mathematics whose concepts demand mental effort, can be impaired.

It must be questioned whether (even older) pupils have sufficient mastery of subject matter, communication, educational goals etc. for really successful long-term work which is independent of a teacher. In an extensive study, Kaiser & Blum (e.g. 1985), together with D. Burghes and N. Green from Exeter and Evesham, England, compared mathematics instruction in England and in Germany. In those areas where the German pupils did better, they identified one cause as the lower degree of direction among English teachers, which entails a lower amount of learning time for the weaker ones among English pupils. (Whether the work with computers can meet this drawback, as is hoped by the CIEC, is not clear.)

(c) There is nothing wrong with social learning, learning to communicate, or taking responsibility, and all over the world schools endeavoured to promote these and similar virtues in the pupils. Only in the last few decades, however, in several Western countries has this been done at the expense of subject matter. This trend seems to be a pedagogical leftover of the 1960s student movement (which itself was part of a great social upheaval in many Western countries at that time). Here the institution of school, the role of the teachers and the structure of subject matter were denounced as parts of the authoritarian capitalist system (Marcuse).

In spite of the computer's military origin and its incessant and ubiquitous use by the military, administration and business worlds and its dominant character towards the users, its potential as a pillar of that authoritarian system was ignored at that time. Despite some deep criticism (mainly not in terms of politics, but of psychology and sociology) many positive qualities were attributed to it such as

- opening up the world of knowledge through its large memory
- making complicated situations and deep concepts accessible through its multimedia potential (in particular: visualizing), or
- making worldwide communication feasible via its net-like linking structure (Internet).

Today, in Western societies and school systems, the computer has the image of relieving the users of old-fashioned restrictions of any kind. It seems as if students cannot help but 'learn', if only the product to be learnt is wrapped up attractively enough (based on the assumption that 'knowledge' is a product to be 'delivered' rather than a process).

One of these restrictions seems to be the necessary, but time-consuming, hard work on mathematics (in general: with effort-demanding subject matter) at school (and in many university disciplines!), which is recognized as a fundamental way of acquiring and securing knowledge. This is a prerequisite for autonomously coping with everyday life as well as for many vocational careers and an essential part of general education. In particular in informatics didactics (= computer science education) a clear trend away from mathematics can be observed, not only reflecting the ongoing rapid change of paradigms in computer science itself, but also as a specialization of modern pedagogy emphasizing general qualifications and neglecting 'hard' subject matter.

The informatics didactician would argue that working on projects, modelling some reality with the computer, experimenting with the model, describing the effects etc. is the subject matter of informatics. Many mathematics didacticians and teachers would also subscribe to this point of view for mathematics as well, at least in a moderate manner. Nevertheless, however the students' activities may be defined, there is a trend to spare them some effort with a lot of 'hard' mathematical concepts and with troublesome teaching & learning methods. There is at least one good reason for this trend. In all countries throughout the world the classical teaching of (advanced) mathematics is quite ineffective (even though at the TIMSS there were countries with rather high scores). There is no evidence however that those pleasant-looking scenarios in which students independently explore computer microworlds, which have some mathematical content, would better meet the demands of society and the concerns of the individual. The improvement in mathematics teaching must proceed from the subject matter (cf. Bender 1998, Wittmann 2000), and the computer can assist in this, as I will point out in the part about geometry teaching.

The CIEC has added one more type of fallacy to those listed above. Overwhelmed by the technical power of the device and by the social change it has already caused, directly and indirectly, and neglecting a lot of other variables, some proponents were (and still are) inveigled into

(viii) Making far-reaching predictions of a psychological, social or pedagogical nature about how computers will change education.

(a) In 1955 the Noble prize winner H.A. Simon predicted that within the next ten years chess computers would beat the best human chess players. Only now, at the end of the millennium, has the computer's level of expertise reached that of the top humans. In the meantime, by adjusting themselves to the computer's strategies, the best human players again managed to obtain better results, thus postponing the computer's final victory for a few more years. It is not the false estimation, by a factor of 5, of the time predicted until computers would play better chess than humans, but of the amount of computer memory and elaborate human-based strategies actually needed, which proves Simon's prophecy to be mere speculation at that time. Notably all this applies to a simple game like chess which has only some two dozen rules.

(b) In his Logo-based educational utopia, the computer scientist S. Papert (1980) predicted the abolition of public schools and their replacement by computers which would be available in every private home. In an interview in 1998 he committed himself to a period "within the next 20 years" (in America? Worldwide?). This prophecy can be taken as a striking example of a computer-based one-sided view on present and future political and social reality (see Bender 1987).

(c) Similarly the German computer educationalist K. Haefner (1982), reviving a catchword from the early 1960s (Picht), predicted a deep educational crisis for the mid 1980s (in Germany?) caused by insufficient use of computers in the educational system. This crisis would only be surmounted by an immense intensification of their use. Every generation of educationalists seems to identify a crisis in the educational system. Possibly it really is in permanent crisis but there is no evidence that this is due to, and can be overcome by, the computer.

(d) With respect to the timing and the scope of the subject matter, the prophecies for mathematics instruction are more moderate, but within the framework of this discipline they sometimes also sound rather radical. Here it is the working mathematician or the aficionado of geometry who is pleased with the possibilities of a Computer Algebra System (CAS) or so-called Dynamic Geometry Software (DGS). They believe that this can and will be adopted more or less literally by the schools and will dominate mathematics teaching (e.g. Hanisch 1992).

Meanwhile, in the computer branch of the German-speaking community of mathematics didacticians, the early enthusiasm of a few has given place to a more cautious approach by the majority (cf. the annual reports of the group 'Mathematics Teaching and Informatics' in the GDM, Hischer 1992ff, and Herget, Weigand & Weth 2000f).

Now, as ever, mathematics is an indispensable element of general education. Traditional mathematics teaching, more or less similar all over the world, has several approved essentials of which the most important is some guidance of the learning process by a teacher. It is true that traditional mathematics instruction is not very effective and that teachers do not really have the learning processes under control. However there is neither evidence nor experience under everyday conditions that autonomous learning in general, and also if based on work with computers, would be at least equally (let alone more) successful. (The phrase of autonomous learning is actually a contradiction in terms as long as there is some specific support, and I question whether, as such, it is desirable, not to mention achievable.)

There have been attempts to re-define mathematics in order to adjust it to working with computers (e.g. Horgan, 1993, on the university level, or Papert, 1980, on the 'school' level), but they gained little success. Indeed new branches emerged (cryptography), old problems were solved ("four colours suffice") and examples and counter-examples could be found and calculated more easily. Each mathematician has become their own typesetter, and at school several skills lost importance (e.g. doing complicated elementary calculations in an automated manner because of the existence of the pocket calculator) or gained importance (e.g. visualizing situations, concepts, relations etc., because of the potential of CAS, DGS etc.). However, in the deepest depths of its substance, mathematics, at all levels, remains the same. This resistance is not due to the inertia of the system of established mathematicians, didacticians and teachers, as several revolutionary representatives of the CIEC suspect; rather it comes from the subject matter itself. Still, the computer is a valuable tool for doing mathematics at all levels and neither the mathematics researcher nor the mathematics teacher can renounce it with a clear conscience.

One of the main desires of contemporary mathematics didactics is the integration of the computer into 'the' mathematics curriculum. In the domain of geometry there is some special research on the stimulation of formal talents like working creatively, independently, collectively etc. (Hölzl 1994, Laborde 1998, Weth 1997 and others) and some practical realization with an accent on subject matter (Elschenbroich 1997, verbal communication by Kadunz, and others), all of them with more or less positive and negative results. In the meantime it is common knowledge that utilizing a spreadsheet, a CAS or a DGS for doing or for learning (by doing) mathematics requires an elaborate body of mathematical understanding, knowledge and skills (which may be further developed with the help of the computer). There are also some pieces of patterns emerging for medium-term mathematics instruction integrating computer and non-computer work (Baumann 1998 and Hole 1998). In Austrian high schools a lot of mathematics teaching is actually done with the use of computers (Wilding 1998, Wurning 1997 and many others). In Germany also there have been (and still are) several more or less isolated projects, often with outstanding teachers and high achievers which use the computer intensively in mathematics teaching (Lehmann, 2001 and many other papers, Thode 2001, and others).

Unfortunately a long term curriculum, say for geometry teaching, is still missing from the first to the tenth (or 13th) grade. I do not call for a direct realization in schools, but for a basis on which, at first, a broad academic discussion and then concrete syllabi, schoolbooks, mid- an short-term lessons, methods etc. can be founded. Geometry has close connections to the rest of mathematics, to many other disciplines and to the real world (including the computer with its many applications) while at the same time it can be treated quite separately. This work would be much more extensive and harder than writing a school book in the conventional way, where one has other school books as a stimulus for good ideas and a long and elaborated tradition of goals, contents and methods (and it would not be pecuniarily lucrative). Of course there have to be the customary differentiations, but now there is one more differentiating element, namely the computer, e.g. in the appearance of a DGS like Cabri Geomètre. (With the permanent 'danger' and chance of a revolutionary further development, one should not cling too narrowly to the specialities of one software package). There have to be alternatives with more or less intensive use of the computer but first of all the goals and then the contents (also mathematically, but mainly epistemologically and psychologically) and the methods have to be re-analyzed in the light of the new possibilities and restrictions. For each teaching unit careful consideration has to be given to which understandings of concepts, knowledge and skills have to be provided, and which are to be developed, how the computer channels this development and how its influence can be made use of. The whole analysis is (I repeat) primarily about subject matter under epistemological, psychological and sociological aspects. As an ideal scenario (ignoring all secondary and tertiary problems and paradigms) one should visualize a classroom with the students being active under the general, and, if necessary, also under the detailed guidance of the teacher.

There are rather early approaches by Schumann (e.g. 1991, 251ff) connecting traditional and computer-oriented geometry and analyzing didactical and methodological questions in detail but proper attention was not given to this work. Instead of exploring the basic ways of understanding and imagining geometrical concepts (cf. Bender 1998) which the students should and/or would adopt, the students were often confronted with rather sophisticated tasks, and their behaviour towards these tasks was investigated (Hölzl 1994, 2000, Laborde 1998 etc.). The analyses of classical geometry didactics, in particular of transformation geometry and the role of (continuous) motions (based on a tradition which is more than 100 years old) (e.g. Bender 1982, Bender 1989, Schwartze 1990), were often not taken into account, although they prove to be highly relevant for the didactics of DGS with its drag mode. Here are some of the old (and a few newer) arguments:

1. Transformation geometry is motionless geometry, as transformations are one-to-one mappings from the plane onto itself without moving any point of the plane. Unfortunately, the notions (e.g. rotation) as well as the usual way of introducing them, namely by moving pieces of paper on a plane or suggesting the imagination of such a motion, leads the learners unavoidably to misconceptions, which, as all experience shows, almost never can be repaired. Here we have a striking example of what Sierpinska (1990) calls 'mental obstacle' and which she thinks is insurmountable.

   But if we leave aside the question of whether students should form a rather abstract algebraic concept of geometric transformation at all or just use transformations as naive tools for investigating geometric figures, the computer can support the formation of that algebraic concept in at least two ways. The pixels which bring about the picture on the computer screen can be viewed as realizations of the points of the plane. It can be made comprehensible for (nearly) everybody that they do not move. By supplying them with colours, parts of them can be united so that they form some (meaningful) shape (e.g. a triangle). By changing the colours in the right way, the illusion can be evoked that the shape moves without the points moving! It is like a wave in the ocean: it is not the water molecules which move from Hawaii to Japan, but their stimulated state of bouncing up and down. Of course, the metaphor of the computer screen can be used without its physical presence, but the students should have some experience with it.

   The next useful attribute of the computer (which has nothing to do with the drag mode of DGS!) is to produce copies of a given shape, which are either congruent with the original shape or distorted following some rule. They can be situated at any required position, without any continuous transformation, thus avoiding the misconception described above.

2. However a large portion of geometric activities has to do with continuous motions or, more general, with continuous changes, and here the DGS can unfold its talents. These activities can be subsumed under the most important general didactical principle of functional thinking which can be formulated as:

   - to aim at structuring problem situations with (mathematical) functions (mappings, transformations) mapping some domain into some range
   - to cruise deliberately through the domain and to observe the effects of this cruising in the range.

   For example: let F be the function which assigns to each polygon its area, and consider all triangles with two fixed vertices A and B. How does the area depend on the third vertex X? Alternatively let R be the composite mapping of two reflections. As one moves from one triangle to another ('move' the triangle), what is the resulting motion of its image (after one, after both reflections)?

   Although continuity of the changes is not necessary, it is very useful in order to recognize the underlying mathematical laws because it is an important means of deriving relationships between the covered positions and thus representing the resulting changes in a concrete manner. Before the advent of DGS, one had to make do with discrete sequences of pictures, and the students had to supply them with continuous transitions (if the teacher wanted them to produce continuous interpretations and mental images).

This is quite different from the idea of the early Geometric Supposer(s) (Schwartz & Yerushalmy 1985), where the computer produced triangles of a kind (say, isosceles) one after the other with the help of a chance generator and there was no claim to any transition between the triangles at all.

Of course the motion of a triangle, produced in a DGS with the drag mode, is also a discrete sequence of still pictures, but it is to be regarded as continuous and it usually is.

3. Seen idealistically, there are mainly two phases of the learning process where the drag mode can be brought into play. Firstly when approaching a new domain, problem, concept, theorem etc., in order to get acquainted with it and to discover, reject or reinforce conjectures. Secondly, after the treatment of a certain domain, problem etc., in order to get an overview, to deepen and secure the insights, to make new discoveries and to make plain the meaning of the new knowledge.

   In my opinion the second phase is the major application of the drag mode by the students (without excluding the first phase or the phase(s) in between). Having themselves a much higher level of knowledge, experience and love for their subject, computer geometry didacticians and teachers tend to over-estimate the students' capability and motivation to put their own questions and answer them self-reliantly with the help of the computer. As nearly all studies show, learners need ample support in the initial phase, and the phrase of discovery learning, allegedly one of the benefits of the work with computers, seems not to be suitable in general. Things look different however, when the learners have some mathematical experience, understanding, knowledge and skills at their disposal.

4. Two major problems of classical geometry didactics and teaching are connected with the concept of mathematical proof. First: a great many assertions in the field of geometry seem to be evident to novices, whereas the expert knows that they have to be proved. Second: how can a novice tell when a 'proof' is a proof? Both problems are exacerbated in computer geometry teaching.

   The computer's image of infallibility, its graphical talents and the reproducibility of some peculiarity (if it is based on a rule) can even more diminish the students' insight into the necessity of proving this peculiarity. In particular, the teacher who stresses the making of discoveries can evoke the idea in the students that the work is done as soon as the discovery is made.

   The question of whether in a geometric situation there is something to be proved, can be regarded as an extreme case of the second question "when is a 'proof' a proof?". — I want to illustrate the extra difficulties connected with DGS with the following examples:

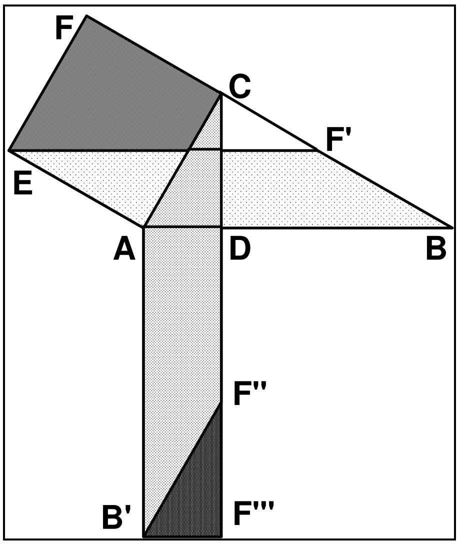

Figure 1: Euclid's Theorem

Figure 1 illustrates Euclid's Theorem. The square AEFC can be mapped onto the parallelogram AEF'B by a shearing along EA, then onto the parallelogram ACF''B' by a rotation by $90°$ with centre A, and finally onto the rectangle ADF'''B' by a shearing along AB'. As all these mappings are area preserving, the original square has the same area as the final rectangle. The three mappings can be represented as continuous motions (respectively deformations) and thus made more vivid, but this does not prove anything. One must know from other conclusions that shearings and rotations preserve area. For shearings there are plausible arguments (for rotations anyway) i.e. during the (continuous) shearing the form gains at the head of the deformation the same amount of area as it loses at the tail. If one considers finite time intervals, this is basically the argument of classical geometry, and it applies immediately to the whole interval i.e. to the transition from the starting state to the ending state, and there is no motion (needed) at all. Introducing the idea of infinitesimal changes either requires very elaborate concepts from calculus which are usually not available at this stage of the curriculum and in addition are not adequate here, or lead to an extremely vague, if not faulty, argument.

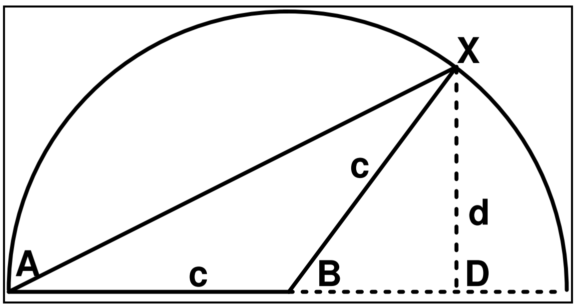

Figure 2: The largest isosceles triangle

Figure 2 (an example by T. Weth) shows (up to congruence classes) all isosceles triangles ABX with their legs AB and BX of fixed length c, the angle at B varying from 0° to 180° (both excluded), i.e. the vertex X moving on a half circle with centre B and radius c . At each position of X (with distance d from the straight line through A and B) the triangle has the area c·d/2 and as c is fixed, this value is maximal, if and only if d is maximal. Either one accepts as being obvious the fact that d is maximal when X is on the 'top' of the circle (B=D), or one must go on with the proof. If D≠B , then there is a triangle BDX with the side BX lying opposite the largest angle (namely the right angle BDX). Thus BX is the largest side, whence d<c . Whereas, if D=B , then d=c .

The assertions used here ('if one angle of a triangle is a right angle, it is the largest' and 'the largest side lies opposite the largest angle') can again be taken as obvious, or ultimately they have to be deduced from a system of axioms. Of course, in modern geometry teaching nobody really works axiomatically. However even if several obvious 'facts' are taken as starting points for geometrical reasoning, it is not only a matter of the students' deficiencies but also a consequence of the curriculum if it is often not clear when a 'proof' is a proof. This is because it is often not clear in principle whether or not a fact is obvious.

Again, on close inspection, the continuous motion of the vertex X does not deliver the proof directly, only indirectly. In order to create continuous motion one looks for a locus of a point, thus reducing the space of solutions from the plane to a line. In the end one has a one-dimensional set of points, of which one (several, or none) has the quality in question, thus representing the solution. There remains, of course, the task of finding the right point whose locus shall be considered, which can make many a geometric problem a difficult one even for experts.

In order to structure the situation, and, in doing so, to get an idea of the position of the solution and hence a proof, it is obvious and reasonable to pass through the locus continuously and directedly, i.e. to use the drag mode, either in one's imagination or with a DGS. This distinction between the (indirect) mathematical and the (direct) didactical function of the drag mode does not always seem to be clear to didacticians and teachers, let alone to students.

Anyway, there remains the crucial problem of translating the conditions of the task into a system of objects of the DGS and of cleverly omitting one of those conditions in order to create a suitable locus as a cognitive basis for the proof.

For more arguments and more examples see Bender (1989).

5. In spite of all the assumed and actual visualizing and stimulating benefits associated with using the drag mode there will be a lot of problems which can be treated better on the basis of discrete sequences of pictures or of just one picture (cf. Lewalter 1997)

   - whether the students' attention can be better drawn to the essential features (if the problem does not consist of finding them)

- whether they are distracted by the impressive visual features of the DGS
- whether they do not feel urged by the drag mode to be active
- whether they are seemingly granted more freedom
- whether they have no choice but do mental geometry
- or geometry teaching includes more content than just pencil-and-paper geometry (PPG) (on the computer screen or on real paper).

6. One of the weak points of DGS is their extensive disconnection from real-world geometry. Some of the main causes of the worldwide decay of geometry teaching in the last century were the treatment of geometry

   - as an axiomatic theory of the Euclidean plane
   - as a range of applications for algebra
   - as a theory of the relations of point sets in the plane
   - as a collection of sophisticated constructions and proofs
   - in short: as pure mathematics.

   All along there has been a broad movement in geometry didactics which has striven for a closer connection between geometry teaching and the real world thus making the teaching more vivid, more application oriented, including more intensively real actions (drawings, handicrafts etc.) and more directed towards the needs of the students. In contrast to this, following nearly all the papers relevant to the subject, DGS just transfers PPG (with many more features, but still disconnected from the real physical world) to the computer screen, thus again laying a much stronger emphasis on PPG. Because of the old reasons (which I cannot discuss here in depth) it is doubtful whether this will re-animate school geometry.

7. There is no doubt that today DGS have to be a part of the geometry curriculum. With their features such as the drag mode, production of loci, the technique of subprograms (modularity), the possibility of continual accurate measuring, arrangement of colours etc., they cover an essential part of that curriculum, namely PPG. At the same time they give rise to basic experiences with New Media, and hence they create some distinguished connections to the real world - not to the physical one, but to the visual one, which is becoming more and more important to all of us.

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