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Students' problems in the identification of subspaces in Linear Algebra

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The goal of the study presented in this paper is the investigation of students' problems with exercises concerning central topics of linear algebra courses at university level. We present the results of our analysis of students' work on an exercise about subspaces of $\mathbb{R}^2$. *We evaluated the written solutions of the task as well as transcripts based on videos taken of student groups working on the problem. We identified and classified descriptions of vector spaces and subspaces that varied widely and demonstrated highly different skills in working with geometric or formal algebraic objects. We analyzed how far students could progress in a complex reasoning process, and identified those steps in the reasoning process on which students needed support to continue.*

Keywords: Linear algebra, vector space, subspace, proof, tutorial groups.

PURPOSE AND BACKGROUND

Problems in teaching and learning of linear algebra have a long history in many countries (Dorier & Sierpinska, 2001). Frequently, the abstract character and the formalism of mathematics that students have not been exposed to in school before is named as a central obstacle (a variety of studies are outlined and evaluated in Dorier, Robert, Robinet & Rogalsiu, 2000). Since vector spaces are a central part and moreover of special importance for almost all disciplines related to mathematics at university, special attention has been paid to them (Dorier, 2000, Stewart, 2017). Generally, students do not develop a clear concept of vectors at school level (Mai, Feudel & Biehler, 2017), and the more abstract approach to this subject taught at universities is described as being "out of reach" by some students (Stewart, 2017). Wawro, Sweeney and Rabin (2011) investigated concept images of subspaces in interviews with students and identified recurring concept images, distinguishing between a subspace as a *part of a whole*, a *geometric object*, and an *algebraic object*. The introduction of first concepts in tutorial meetings in linear algebra, with a special focus on the behavior and influence by the tutor, has been studied by Grenier-Boley (2014).

CONTEXT AND DESIGN OF THE STUDY

In this study, we investigated the problems of students shortly after their first encounter with vector spaces and subspaces at university level. The participants of our study were students with major mathematics or computer science, enrolled for bachelor of science or bachelor of education (for secondary school, "Gymnasium"), most of them in their first semester. In our study, we collected data from students working on tasks in groups during their tutorial group meetings (1.5 hours), where the tutors were advised to answer questions but to only intervene when the students had substantial problems to continue. The students worked on exercises about the content of a recent lecture under the supervision of a tutor. In this context, we assigned special tasks that we developed ourselves together with the lecturer and his assistant, but we did not influence the course design otherwise. We will report only on one of them in this paper. The course can be considered to be typical for a beginners' lecture in linear algebra, which normally is rather abstract, and was given by an experienced lecturer. During their tutor meetings, the students worked on our exercises on separate sheets that we collected, scanned for later analysis and gave back to the students in the next meeting without any grading or corrections. We gathered between 78 and 130 written works on each exercise. Moreover, we also took video recordings of groups of 2–4 volunteering students working on these exercises. They worked on the exercise under the supervision of a student tutor who was part of the research team and familiar with our a priori analysis of the task. The experienced tutor was advised to help the students if they struggle with the exercise in the same way as she would do in an ordinary tutor group meeting. We were interested in identifying important didactic variables. The results obtained by analyzing the first implementation of the exercise about vector spaces are currently being used for designing a second implementation in the course Linear Algebra I.

The task for students in our study and preliminary research questions

In this paper, we concentrate on an exercise about subspaces and vector spaces (see figure 1) that was part of the exercise sheet during week 7 of the course, immediately after the notion of subspaces had been introduced. Students are taught analytical geometry and linear algebra at school level, where vectors are introduced as tuples (or classes of arrows), but they do not as a rule have a clear concept of a vector (cf. Mai, Feudel & Biehler, 2017). Students know equations of planes and lines in $\mathbb{R}^2$ and $\mathbb{R}^3$, without considering them as subspaces, because this notion is not taught at school level.

The following exercise was designed in order to provide two different kinds of learning potentials (as described in Gravesen, Grønbæk and Winsløw, 2016):

1. Linkage potential: In part a) to e), our intension was to motivate the students to activate their school knowledge concerning the description of geometric objects using equations; we hoped that they would recognize the sets as descriptions of lines, points, parabolas etc., and connect this knowledge with the new concepts of vector spaces and subspaces.
2. Research potential: Part f) of the exercise was created in order to engage the students in a research-like activity. Even if achieving a complete solution seemed unrealistic for most of them, we were interested in how the students would approach this open question. They had to formulate a hypothesis and use abstraction to identify and construct subspaces. The exercise can be seen

as a “mini research project” that differs in type from standard exercises.

Exercise Which of the subsets of $\mathbb{R}^2$ given in part a) to e) are vector spaces with respect to the addition and scalar multiplication defined on $\mathbb{R}^2$?

a) $M_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + 2x_2 = 0\}$

b) $M_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + 2x_2 = 1\}$

c) $M_3 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = -2, x_2 = -1\}$

d) $M_4 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2^2 = 0\}$

e) $M_5 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \leq 0\}$

f) Try to find all subspaces of $\mathbb{R}^2$. Make it clear to yourself that you indeed found *all* subspaces. A formal proof is not necessary.

Figure 1: Exercise on subspaces (translated from German)

The parts a) to e) can be solved by a formal check whether the properties of subspaces are satisfied by the provided sets. As this was learned in the previous lecture, this is a standard task. Geometric ideas are not necessary, but we hoped that students may do geometric interpretations of the sets to develop a geometric meaning of subspaces and non-subspaces of $\mathbb{R}^2$. Task f) is different, because this is the first time that this type of question is asked. Students may use the results from a) to e), that have provided examples and counterexamples of subspaces, to find the zero space, all lines through the origin, and the whole $\mathbb{R}^2$ as subspaces and give reasons why they are subspaces on some level. The challenging question is whether or why these are *all* subspaces of $\mathbb{R}^2$. Research questions concerning f) are: How many students identify the zero space and the entire $\mathbb{R}^2$ as subspaces? Are all lines through the origin identified as subspaces? Which arguments do students provide for considering a line through the origin as a subspace of $\mathbb{R}^2$? How do they reason, when exploring, whether there are more subspaces in $\mathbb{R}^2$ or whether they have already found all?

We were also interested in the sources of knowledge students used, such as their results on a) to e), parts of the lecture, or geometric interpretations related to their school knowledge, and concerning the videographed tutorial sessions, which kind of support by the tutor they can use in their reasoning process.

METHODOLOGY AND DATA COLLECTION

For the analysis of the written work of the students, we followed the method of Biehler, Kortemeyer and Schaper (2015), by comparing each solution with the so-called *student expert solution* (SES), which is a sample solution based on the idealized actual knowledge of the students at this point of the lecture. Moreover, the student expert solution contains additional meta-information about the solution, for example, several alternative opportunities for solutions and explicitly written-out learning objectives. In relation to Brousseau’s theory of didactic situations, this method can be seen as a special approach to the development of an *a priori analysis*. We evaluated the written work in a two-step procedure: In a first step, we categorized the solutions by correctness and collected peculiarities and mistakes. Based on this and the *SES*, we developed a detailed coding system for deeper analysis. The recorded videos have been transcribed in order to allow a detailed qualitative analysis.

A PRIORI ANALYSIS OF THE TASK

In the lecture, the definition of vector spaces was given in a typical traditional, abstract way. The zero space and the vector spaces K (trivial vector space over K) and $\mathbb{C}$ (the latter together with component-wise addition and multiplication) had been presented by the lecturer as first examples. Apart from this introduction, the students had only seen the following (relatively abstract) non-trivial examples for vector spaces in the lecture: (VS1): K^{n}, the "standard vector space", where K is any field, with n ∈ ℕ, including the definition of addition and the scalar multiplication (component-wise), (VS2): $K^{\mathbb{N}}$, the vector space of sequences over the field K, with the component-wise operations. Subsequently, subspaces of vector spaces had been defined to be subsets of vectors spaces that are vector spaces themselves with respect to the same operations. Following this, they had learned that a sufficient criterion for proving that a nonempty subset W of a vector space V over the field K is a subspace is to prove that ly $\forall\, v, w \in W \Longrightarrow v + w \in W$, and secondly $\forall\, v \in W, \forall\, a \in K \Longrightarrow av \in W$. As examples of subspaces, the trivial subspaces $\{0\}$ and V were nominated without proof. Moreover, for both vector spaces (VS1) and (VS2), there was an abstract example for a subspace given, and we state the first one of them here since it will be of use for our later analysis:

(S1) The set $L := \{(x_1, \dots, x_n) \in K^n \mid \forall\, 1 \leq i \leq m\colon \sum_{j=1}^{n} a_{ij}x_j = 0\} \subseteq K^n\}$ is the solution space of a homogeneous linear equation system $\sum_{j=1}^{n} a_{ij}x_j = 0$. It was shown that this set is indeed closed with respect to addition and scalar multiplication and is a subspace. Note that this example can be applied to $\mathbb{R}^2$, if we choose $m = 1$. The subspaces in L are the lines through the origin expressed by linear equations. This interpretation could be done by students on the basis of school knowledge. The lecturer did not provide this specialization himself.

For our later analysis, the following distinction is central. All provided examples have in common that sets are characterized by equations (subspaces defined by *relations)*. In contrast, 1-dimensional subspaces could also be defined by *explicit construction:* for instance for any $x \in V$: $L_x := \{v \in V | v = \lambda x, \lambda \in K\}$. The latter way of defining subspaces was not yet a topic of the lecture, which will turn out to be an obstacle for some students. Moreover, the students had not seen any geometric interpretation or visualizations of vector spaces or subspaces, in particular no (concrete) examples of subspaces in $\mathbb{R}^n$. In the following, we will give an overview about possible approaches and steps to part f).

Step 1: Find some subspaces. With the knowledge from the lecture, the trivial subspaces (the zero space and $\mathbb{R}^2$) can be named. To find nontrivial subspaces, one can identify again the set M_1 of the previous part a) of the exercise as a subspace. Starting with this set, one could generalize from numbers (like 2 and 1, as chosen in part a)) to a general form with coefficients, and give the set $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$, with $a, b \in \mathbb{R}$ not both being zero. Instead, if one abstracts from the mathematical language used in the exercise before, these sets could also be expressed constructively as $L_x \coloneqq \{v \in V | v = \lambda x, \lambda \in K\}$. Supported by Dorier, Robert, Robinet and Rogalsiu (2000), we expected difficulties to translate the relational representation into the constructive representation and vice versa. Alternatively, with the knowledge from the lecture, one could apply the example (S1) given in the lecture to the space $\mathbb{R}^2$, and describe the subspaces in terms of the solutions of homogeneous linear equation systems. This reasoning can be done just algebraically. It could also happen that students use geometrical terminology concerning lines through the origin.

Step 2: Verification of the subspace properties. In order to reason why the subsets given in step 1 are subspaces, one could either refer to the solution of part a) of the exercise or (for the trivial subspaces and in case of the use of the solution spaces of homogeneous linear equation systems) to the lecture. In case of a geometric description ("lines through the origin"), either geometric or algebraic arguments have to be provided to verify the subspace properties.

Step 3: Why are these all subspaces? The final challenge is to reason if and why all subspaces of $\mathbb{R}^2$ have been found. This can be done algebraically, but we did not expect our students to complete this reasoning process in the given time, since it requires a development of several successive algebraic arguments. Based on their school knowledge, the students could recognize the descriptions of geometric objects by equations in part a) to e) and abstract from the previous results, leading to the conclusion that lines through the origin are subspaces, but no other lines, single points or other collections of points. At this point, a successful reasoning based on school knowledge could be done constructively, based on geometric arguments. Trying to construct "bigger" subspaces than just the lines through the origin, a student could build the union of two different lines and check whether this set is a subspace. Alternatively, he or she could try to find the minimal subspace that includes one line g_0 through the origin and an additional point x_0 not lying on this line. He or she could come to the conclusion that this has to be the whole $\mathbb{R}^2$. A formal argumentation here is that every point can be represented as a linear combination of a point $x_1 \neq (0,0)$ from the line g_0 and x_0, but even if the student does not come to this conclusion at this point, he or she could have the idea to consider the line through the new point x_0 together with the original line, and therefore check this new set for the subspace conditions. He or she could check the closure of addition or come to the idea that further points have to be added to the union in order to get a subspace. Since this type of reasoning seemed to us more likely to be achieved with the previous (including school) knowledge of the students, the tutors in the normal tutor group meetings as well as the tutor in the video study were advised to guide the students along this reasoning process if they struggled in approaching the problem. Based on this sample solution process description, we tried to answer the following research questions in our analysis:

1. How far in this three-step process would the students come when they work on this exercise? Would they even be aware of the need to do step 2 and 3?
2. Would they favor one of the described approaches to the problem (geometric, algebraic), and would they use the constructive or the relational way to describe the 1-dimensional subspaces? Would they approach step 3 in a constructive way, building up subspaces starting with just one point, as described above, or would they find other ways (purely algebraic?)?
3. Finally: Would they recognize that parts of exercise f) could be solved by an application of the example (S1) given in the lecture?

Since we posed the question in part f) in a relatively weak phrasing, we could not expect the majority of students to give a fully structured, formal reasoning in this exercise, in particular for the steps 2 and 3. But we were interested if the exercise itself would stimulate the students to give reasons for their answers and, in particular, how they would argue in this case.

RESULTS

To find answers to our questions, we analyzed the written works as well as the video recordings of the students working on part f).

Work on part f): Written exercises

From the written works of 116 students on this exercise, just 48 handed in solutions for part f). This is most likely due to the fact that the time was very limited, so many students just did not come to part f). We analyzed their work with respect to the three steps of the solution as described in the a priori analysis.

Trivial subspaces	$\mathbb{R}^2$	33
	Zero Space	32
1-dimensional subspaces	Solution with any description of the 1-dimensional subspaces (some students used more than one description)	33
	- Relational description: $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$	24
	- Constructive description: $L_x := \{v \in V \mid v = \lambda x, \lambda \in K\}$	4
	- Geometric descriptions: “line containing zero”, “line through origin”	12

Table 1: Frequency of the nominations and descriptions of the subspaces

Step 1: Which subspaces do they find? How do they describe them? Do they use previous parts of the exercise or name the set considered in part a)?

The results in table 1 were collected by counting how often the three types of subspaces were mentioned in the solutions. Hereby, each notion of a subspace counted, as long as it was clear enough to denote the required set. How did the students describe the 1-dimensional subspaces? We distinguished between “geometric” descriptions, using expressions like “line containing zero”, “line through the origin”, relational descriptions using a set like $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$ or something mathematically equivalent (see a priori analysis for a definition of this category) or constructive descriptions like $L_x \coloneqq \{v \in V | v = \lambda x, \lambda \in K\}$. Some students used more than one description in their solution. Apart from this, it was interesting to see that only 8 students did mention any part (mostly a)) of the previous exercise in part f). It is not clear if those who could not give any (nontrivial) subspace actually never recognized that the set M_1 from part a) is a subspace (since the word “subspace” was not used in part a)), or if they just forgot about it before they started with part f). Moreover, it is interesting that the trivial subspaces, which we expected the easiest to find, were not nominated more often than the 1-dimensional subspaces. We were also surprised to see that only 2 of the students did refer to example (S1) (see a priori analysis) from the lecture, concerning the solution spaces of homogenous systems of linear equations.

	No reasoning	Incorrect reasoning/ unclear approach	Partial reasoning	Complete reasoning
Step 2	34	5	7	2
Step 3	35	6	6	1

Table 2: Frequency of reasoning in part f)

Step 2: Do they show that the given sets are subspaces? How do they argue?

Most students did not give reasons (see table 2 for results), but within those who did, we distinguished between approaches that did not go in the right direction (for instance students just answered by listing all properties of a subspace without proving them or claimed that it was “clear” that the spaces are subspaces), students who did give a correct approach or a partial proof (they mentioned that closure must be proved, but did not, or just checked the addition or the scalar multiplication, or just checked an example etc.) and complete solutions with full reasoning (using example (S1) from the lecture in both cases).

Step 3: How do they reason that they found all subspaces of $\mathbb{R}^2$*?*

Within the 13 solutions that had some kind of reasoning (see table 2 for results), we distinguished again between unclear or vague approaches to reason the completeness of the given list of subspaces (for instance the statement “there are no other possible, because one cannot multiply vectors”), promising but incomplete approaches (some students gave reasons why lines not going through the origin cannot be subspaces, but did not consider other subsets, or just discussed the closure of one of the operations) and complete reasoning (just one case, again applying example (S1) from the lecture).

Work on part f): Video recordings

We give a summary about three groups of students that we recorded during their solution process on part f) under the perspective of our research questions. Due to limitations of space, we cannot document the method used to analyze the transcripts and the students' interaction in more detail.

Group 1: The first group tried to find subspaces by systematically going through the list of properties, and found the zero space to fulfill them. Then, they remembered the set proven to be a vector space in part a) and generalized it to a set of the form $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$ after a discussion whether the coefficients are arbitrarily exchangeable without harming the subspace conditions. They discussed the closure of the vector space operations in this set, but referring to the proof they had given in part a), they convinced themselves quickly that there was nothing else to prove. After this, they also identified the full space $\mathbb{R}^2$ since there is no claim for a subspace to be a proper subset. At this point, they were asked by the tutor if and why they found all subspaces now. They had the idea to consider the set $\mathbb{Z}^2$ and, referring to their knowledge about groups, discussed the closure of operations on this set before they could finally rule it out to be a subspace by the fact that the scalar multiplication with elements from $\mathbb{R}$ is not closed on $\mathbb{Z}$. The tutor then asked them to consider the set $M_{a,b}$ geometrically. They start to consider the tuples of coefficients (a, b) in the plane instead of the equation $ax_1 + bx_2 = 0$. With another hint from the tutor, they found out that the set $M_{a,b}$ whose elements are described by the equation $ax_1 + bx_2 = 0$ denotes lines in the plane, and discussed the closure of the operations for these lines. The students did not develop an idea themselves to give arguments why they had found all subspaces. However, the students were able to follow the geometric constructive reasoning of the tutor (see a priori analysis).

Group 2: The second group came up with the idea to apply example (S1) from the lecture. After some discussion and a bit help from the tutor, they found that the subspaces defined there are the solutions of one or more linear equations, each having two coefficients. The central difficulty for them was to see that the number of coefficients is fixed to 2, but there could be an arbitrary number m of equations in a system of linear equations that is still defined in $\mathbb{R}^2$. It was a real discovery later that $m = 1$ provided descriptions as are provided in $M_{a,b}$. Up to this point, they did not consider the trivial subspaces at all. They struggled a bit to write down the concrete subspaces they could find this way in terms of algebraic expressions, but managed it with some help from the tutor. Asked whether they found all subspaces, they did not develop the idea to consider the spaces as lines in the plane on their own, but after the tutor came up with this idea, they were able to work with this concept after a short phase of orientation in which they convinced themselves that the geometric objects stand for the same subspaces they worked with before. Just at this point, they identified the trivial subspaces too. Step 3 (to reason that all subspaces were found) was only solved with tutorial support (similar to group 1).

Group 3: The third group used the previous parts a) to e) in their reasoning and started with the subspace found in a), but immediately identified this set to describe a “line through the origin”, which gave birth to a generalization to all lines through the origin. They continued to orally communicate in geometric terms, but decided to write down the set using the relational algebraic expression $M_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 = 0\}$. They named the trivial subspaces without further comments. The proof given in part a) sufficed for them for a reasoning of step 2, and they started to discuss step 3 quickly. They used references to part b) to e) to rule out other types of possible subspaces. The group thought they had finished at this point. It was the tutor who pointed out that step 3 was not yet satisfactorily answered. Different from the other groups, they took up the tutors input to construct other subspaces geometrically and in order to find out that such subspaces have to be equal to $\mathbb{R}^2$. With some minor help from the tutor, they finished this step quickly, needing much less time for the full task than all other groups.

CONCLUSION AND DISCUSSION

As a result, we can state that students at this point in their studies were able to find and describe (using varying descriptions) subspaces of $\mathbb{R}^2$, but the question to find all subspaces was a serious obstacle for the students. Moreover, the step to translate the algebraic description of the subspaces into a geometric view, where reasoning could be done with less formality, was a further obstacle for them, since they seemed not to connect or apply their geometric knowledge from school to the new problem.

It seems like a geometric approach to this kind of problems is not a natural, automatic behavior of students at this point of their education. This result resonates with the observations from Wawro et al. (2011), who stated that intuitive geometric notions can be the preferred approach of first year students to the concepts of subspaces, but also cause problems if they their geometric intuitions are inconsistent with the formal definition. It is worth pointing out that our students did not, in opposition to the results of Wawro et al., automatically identify (often mistakenly, if there was no respect to a necessary embedding) the $\mathbb{R}^1$ as a subspace of the $\mathbb{R}^2$. A possible explanation for this result is the fact that our results were obtained shortly after the introduction of subspaces in the lecture, where Wawro et al. interviewed their students when they already have had more time to develop a concept image of subspaces, including some misconceptions.

Most students did not connect the different parts of the exercise, appearing in different “languages” (like the sets in the parts a) to e) and the open question in part f)) to solve the problem in f). With some help, especially with the request to consider the sets geometrically, they were able at least to understand reasoning on this basis, and some students actually could even give proofs or approaches to proofs on their own. We came to the result that the students needed more guidance and preparation to solve this problem, and in particular support that helps them to deal with each step and even sub-step of the solution of the problem in part f). In our subsequent study in winter term 2017/2018, we are investigating if explicit indications in a) to e) to consider the sets geometrically and a rephrasing of part f), splitting it up into more explicitly described steps, have a decisive influence on the students' ability to solve task f).

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