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Plenary paper

The transition from Calculus and to Analysis – Conceptual analyses and supporting steps for students

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Abstract

The paper starts with discussing the transition problem by characterizing the type of mathematics that is characteristic of school calculus. General types of measures for supporting students in the transition phase will be briefly reviewed. An essential aspect of the transition problem is the new culture of mathematics that is underlying Analysis courses. Arguments for making this change in culture more explicit and some concrete suggestions will be provided. The paper discusses examples from two empirical studies to support this analysis. In the first study, some results of an Analysis 1 final examination course in the first semester are analyzed. One task of the examination is taken, where school-mathematical solution strategies conflict with university-based norms. A second example is taken from a design-based research study, where a workshop was designed for supporting the guided reinvention of the concept of convergence of a sequence before formally introducing it in the lecture. One lesson learned of this second study is to be much more explicit about which conditions mathematical definitions at university have to fulfil, which seldom is a topic of explicit instruction.

Introduction

Students in Germany obligatorily learn Calculus in the last years of the college-bound schools leading to the Abitur (Gymnasium) and usually take courses on Analysis in their first year of studies if they have chosen a subject that has a mathematical component. These Analysis courses vary depending on whether the students have entered a programme for mathematics majors or, for instance, for future engineers or economists. In general, the university courses on Analysis in Germany are similar in mathematical style to the Real Analysis courses in the US, and the calculus courses at colleges in the US are somewhat "in-between" the school calculus courses in Germany and the University Analysis courses. In other words, the transition from Calculus to Analysis in Germany coincides with the secondary-tertiary transition.

Calculus at the school level: sources of transition problems

The relatively new German national standards for the last three years before the Abitur include standards for teaching calculus (KMK, 2012). However, it is certainly not wrong to say that these standards do not focus on easing the transition to Analysis at the university level. The transition is relevant for a substantial part of students, namely those who study STEM subjects or economy. However, these students are not the majority. Calculus teaching has to provide an excellent education for a broader audience, where aspects of general education (Allgemeinbildung) have to be taken into account (Biehler, 2019). A significant concern of the mathematics education community is that students do not just develop calculation skills at the school level but also the conceptual understanding of the basic concepts. The notion of "Grundvorstellungen" ("basic mental models" is an approximate translation) is used to characterize this understanding. This notion is related to the notions of mental models and concept image (Greefrath, Oldenburg, Siller, Ulm, & Weigand, 2016). Derivative should be understood as the local rate of change, the slope of the tangent, or the best linear approximation. The cumulation aspect of the integral including reconstructing the "stock function" from the rate of change function should be important conceptual ideas, besides understanding the definite integral as a measure for the oriented area under a function graph. These "Grundvorstellungen" are particularly necessary for the effective use of integrals and derivatives in modelling contexts. The theoretical aspects of calculus lie not in the centre of a typical course.

Recently, there have been efforts to offer non-compulsory additional courses already at the school level to students who are interested in entering study programs in STEM subjects. This trend is in line with recent recommendation of the German associations of mathematics, didactics of mathematics and STEM teachers[1] who also recommend measures for the first semester of university studies such as pre-university bridging courses and renewed curricula in the first study year. These recommendations are backed up by a recent Delphi study in which more than 800 German mathematics university teachers were asked, what they consider as necessary prerequisites for students entering a STEM university program (Neumann, Heinze, & Pigge, 2017; Pigge, Neumann, & Heinze, 2017).

A recent book that stems from such a course offered at school level is the book by Proß and Imkamp (2018), which – in large parts – focusses on the transition between Calculus and Analysis. Similar content is also covered in university-based pre-university bridging courses such as the studiVEMINT course (http://go.upb.de/studivemint), where the author of this paper is a co-developer of. I will take the book by Proß and Imkamp (2018) and the studiVEMINT course as an example of a view of what topics and competencies are currently missing in school mathematics for easing the transition. The courses improve the "technical basis" for Analysis at the university level in that covers algebraic techniques such as inequalities including those with the absolute value function, and equations with powers, roots and logarithms and of course practices to work with algebraic terms (including those with fractions). However, this is not a new type of mathematics. These topics "disappeared" from school mathematics in recent years.

In contrast to textbooks at the school level, the presented rules are mathematically explained and justified if not proven. In the terminology of ATD (the Anthropological Theory of Didactics) the technological block is expanded. However, different from university courses (Winsløw & Grønbæk, 2014), the new technology in the sense of ATD does not become a practice on a higher level: students are supposed to fluently calculate but not to prove theorems about roots and logarithms.

A different approach elaborates the function concept through emphasizing related concepts such as injectivity, surjectivity, composition of functions and inverse functions are treated as well as domain and co-domain emphasizing functions as mappings. This perspective on functions is different from the current perspective in school mathematics. We also find a chapter on sequences and limits, including its formal definitions. The topic "sequences and limits" has practically disappeared from

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[1] http://mathematik-schule-hochschule.de/images/Massnahmenkatalog_DMV_GDM_MNU.pdf

the secondary curriculum. Often, limits are introduced the first time when the derivative concept is introduced, but not as a separate chapter. If school textbooks have separate chapters on sequences and limits, these chapters are often non-compulsory, and sequences and limits are – if at all - introduced for modelling discrete processes of real-life situations but not as fundamental theoretical means for the conceptual development of calculus. The above book also contains a chapter on continuity and a section on differentiability. Both concepts do not play a significant role in secondary schools. Continuity usually has no separate chapter in a school textbook. Continuity is sometimes mentioned as a graphical property of functions ("one can draw the graph by a pencil without jumps and holes"). As the fundamental theorem of calculus (FTC) is a compulsory topic in secondary mathematics – all students should know a visual proof of that theorem according to the standards. The intuitive concept of continuity is mentioned as an assumption, but it is rarely pointed out, why this assumption is essential, and it is often quickly forgotten by students because the most important "meaning" of the FTC at school level is the justification for determining values of definite integrals by using primitive functions. This meaning is represented in the practical block surrounding the FTC. Proß and Imkamp (2018) and the studiVEMINT bridging course also discuss differentiability and non-differentiability of a function as a conceptual prerequisite for defining the derivative, whereas most school textbooks do not question the existence of a derivative, which is correct for nearly all the functions in a school calculus course. There are some notable exceptions, however, which are downplayed in most school textbooks. The integral is often introduced by starting with step functions in a modelling context, where the oriented area under a step function can be given a meaning in the context. For instance, if the step function can be interpreted as velocity, the area function stands for the covered distance. The area function depending on a variable x can be calculated using elementary geometry. The resulting area function has points of non-differentiability where the step function changes its value (where it has points of discontinuity).

These differences are discussed in teaching material for courses in didactics of calculus (Biehler, 2018) and based on an analysis of selected textbooks. A systematic textbook analysis would undoubtedly confirm the compartmentalization structure of school calculus focussing on separated practical blocks similar to what analyses of other topics from the perspective of ATD have revealed (González-Martín, Giraldo, & Souto, 2013). This structure of school calculus is fundamentally different from university analysis. An even deeper lying epistemological difference was brought into focus based on textbook analyses by Witzke and colleagues (Witzke, 2014; Witzke & Spies, 2016) who characterize school calculus as an empirical theory of graphically represented function graphs. This characterization is related to the fundamentally different role graphs play in school Calculus and in university Analysis (Weber & Mejía-Ramos, 2019).

Transition problems in the first study year

In general, at the university level in Germany, we can observe three types of measures for supporting students in the transitional phase. In the WiGeMath project, we are collaborating with 14 German universities with regard to analysing and evaluating support measures. We analyse the most common types of measures: pre-university bridging courses, so-called bridging lectures for the first study year that are especially designed for easing the transition, and measures that accompany traditional lectures such as the creation of mathematics support centres, non-compulsory additional tutorials, or supportive e-learning material (Kuklinski et al., 2018; Liebendörfer et al., 2017). What is not so common in Germany is a fundamental content redesign of the standard introductory lectures, i.e. Analysis and Linear Algebra.

A wide-spread measure in German universities is to offer pre-mathematical bridging courses for future students to support the transition process. Universities offer these courses in the months between attending school and starting at university, and they last between two and six weeks. Calculus and Analysis play a role in most of these courses, goals and orientations of these courses vary considerably (Biehler & Hochmuth, 2017; Biehler et al., 2018) Some courses repeat school mathematics, and others introduce mathematical concepts at a level of rigour similar to later university mathematics. The latter type of bridging course aims at preparing students for the upcoming level of rigour and a changed mathematical practice or praxeology. Among others, they already practice the new role for formal definitions and theorems, and the new role of arguments based on graphical representations, which are no longer accepted as valid arguments but only as preparatory "heuristic" ones. Some courses make the transition into a new practice with different features explicit instead of just working on one of the levels. In these types of courses, the conscious change of the expectations and the mathematical belief systems of the students stand in the focus while taking their previous knowledge and orientations into account.

Felix Klein is often quoted speaking about the “double discontinuity” of the transitions that mathematics teachers have to face: school mathematics -> university mathematics -> school mathematics (Winsløw & Grønbæk, 2014). We are concerned with the first transition in this paper, but for the second transition in calculus, see Wasserman, Weber, Fukawa-Connelly, and McGuffey (2019).

The new praxeology of university mathematics is – as a rule – not an explicit topic of teaching in the standard first semester courses such as Analysis. The new style of defining, proving and theory development is practised and communicated through examples in the lecture. A decisive role for the enculturation has the feedback that students receive to their written homework, either as written comments to their submitted work or orally in weekly tutorial sessions, where norms for adequate proofs and written argumentations are discussed at least implicitly. Part of the students does not understand and master this largely implicit enculturation. In a joint project with Leander Kempen (Kempen & Biehler, 2019a, 2019b), we newly designed a first-semester course called "Introduction into the culture of mathematics". The accompanying research followed a design-based research paradigm, and we re-designed the course three times based on the previous research results. One of the significant changes was that we became more and more explicit about the question of what counts as proof and what is an adequate use of formal mathematical language. We used the language of socio-mathematical norms (Yackel & Cobb, 1996) for the analysis and redesign of the course. A redesign of an Analysis 1 course may profit from this approach.

In the following, examples from two empirical studies done at the khdm (competence centre for research in university mathematics education, www.khdm.de) are presented to support this analysis. In the first study, the results of an Analysis 1 final examination at the end of the first semester are analyzed. One task of the examination is taken, where school-mathematical solution strategies conflict with university-based norms. A second example is taken from a design-based research study, where a workshop was designed for supporting the guided reinvention of the concept of convergence of a sequence before formally introducing it in the lecture. One lesson learned from this second study is to be much more explicit about which conditions mathematical definitions at university have to fulfil, which seldom is a topic of explicit instruction.

Conflicts between school mathematics and university mathematics education: An example from a final Analysis 1 examination

Enculturation can be less successful than expected. I will pick up an examination question of the final examination of a first-semester Analysis course for mathematics majors and future Gymnasium teachers. A systematic analysis of students results in the complete examination will be published elsewhere.

The task from the examination is shown in Figure 1.

Aufgabe 7 (my translation)
Consider the Function $F: \mathbb{R} \to \mathbb{R}, x \mapsto \int_1^x t^3 \exp(t)\, dt$.

(a) Prove that F is differentiable and $F'(x) = x^3 \exp(x)$. (4 points)
(b) Determine all local minima and all local maxima of F. Determine the most extended possible intervals in which F is monotonically increasing and the most extended possible intervals in which it is monotonically decreasing. (8 points)

Figure 1: The examination task

The task is very close to school mathematical tasks but also has essential differences. The results were surprisingly bad from the perspective of the lecturer and the researchers. The average number of points was 4.14 of a maximum of 12 possible points (see Figure 2). The written test had nine tasks; five of them had better average results than task 7.

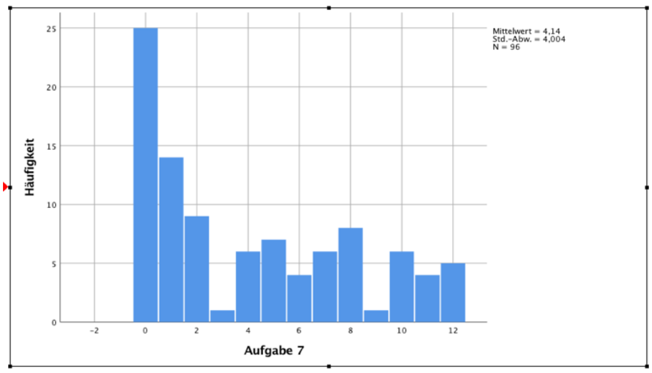

Figure 2: Distribution of points of the examination task (N = 96), mean no of points: 4.14 of 12)

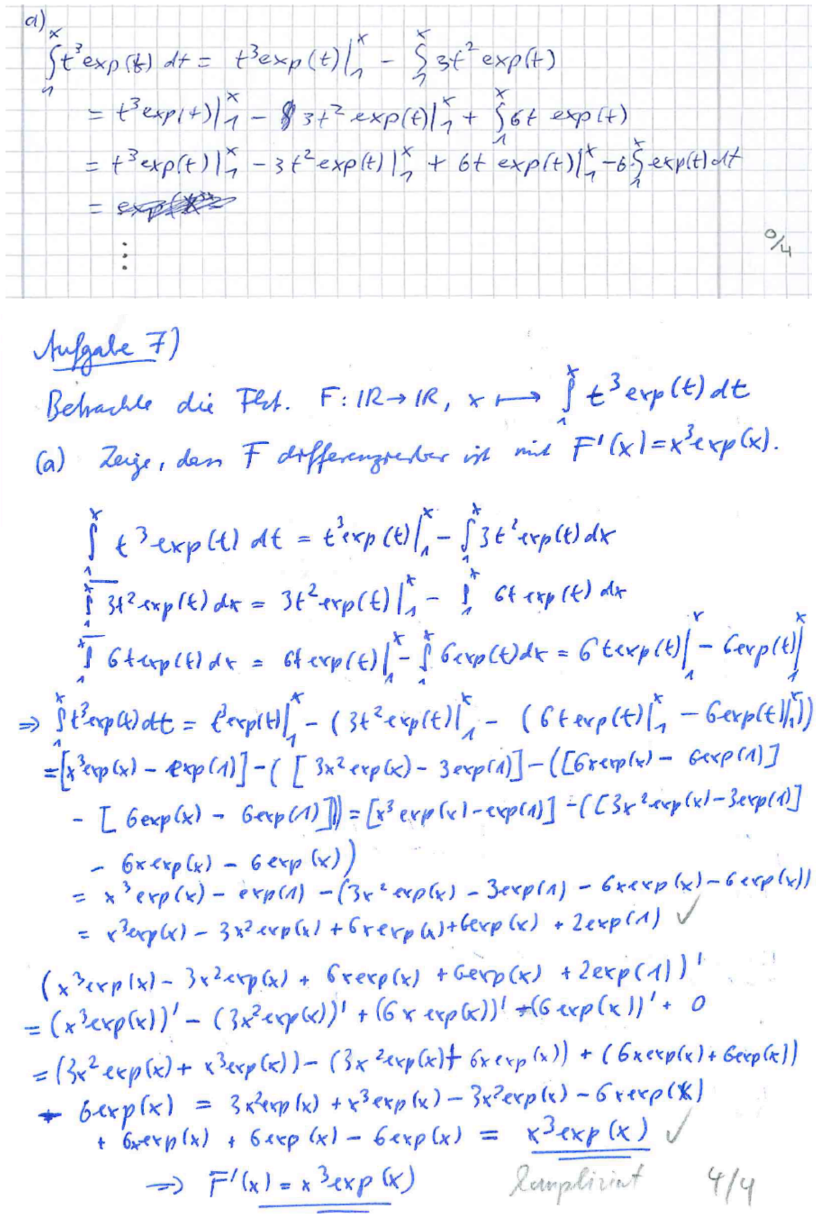

Figure 3: Two example solution for calculating the integral and then the derivative, one of them with success.

Subtask a) is a trivial one if one has understood the theoretical claim of the fundamental theorem of calculus. However, most students who worked on the task tried to calculate the integral (often not successful, see Figure 3 for the first example), and then they tried to calculate the derivative. The task a) would not have been posed in school mathematics, where the notion of differentiability is downplayed if at all mentioned and the theoretical nature of the FTC is downplayed in favour of its practical value to calculate integrals. One interpretation of this comparably unfortunate result is that students did view the FTC still from a school mathematical perspective and interpreted the task just as a calculation task.

Subtask b) is largely a school mathematics task, which, however, has to be solved on the level of university mathematics. Calculations show that F'(x) = 0 iff x= 0, but F''(0) = 0. In order to argue that there is a local minimum, students could reason with the sign change of F`(x) in x = 0, or calculate further derivatives finding that the first derivative with $F^{(n)}(0) \neq 0$ is with n = 4, where $F^{(4)}(0) > 0$. This result can be used as an argument that there is a minimum of F in x = 0. However, these conditions based on higher derivatives are often not part of the university course. Our students had difficulties in various respects. For instance, F''(0) = 0 is wrongly interpreted in that there is no extremum in x = 0 (that this is a wrong conclusion is discussed in most school textbooks already). Another source of problems is that the statement "F'(x) = 0 iff x = 0" is often not interpreted as a theoretical result stating that can be no other extrema beside x = 0. In current school teaching, the condition F'(x) = 0 mainly has the practical function to enable the calculation of extremal points. This tradition is challenged – but has probably not yet fundamentally changed -- by the widespread introduction of graphical calculators. If one "sees" a minimum in a function graph in a certain window and this seeing is allowed as an argument in students reasoning, it becomes more important to know that further extremal points cannot exist theoretically. So, graphical calculators have the potential to show the need for theoretical argumentation. However, these tools are not introduced in schools for this purpose.

Anyway, let us focus on the university context again. Explicit use of the theoretical nature of "F'(x) = 0 iff x = 0" in the argumentation, however, is necessary to give a complete answer to the question to find **all** extremal points.

We can only speculate what changes in the teaching of the Analysis course may lead to better results

(a) The distinction of the theoretical and the practical value (for calculations) of a theorem has to be explicitly introduced.

(b) In cases where university mathematics overlaps with school mathematics (such as in the case of finding extremal points), it may pay to explicitly contrast and relate the school mathematical reasoning to the university mathematical reasoning. This would help students see school mathematics from a higher standpoint, concretized by examples and not just as lecturers' lip-service to globally justify why Analysis is different from Calculus. With such examples, theoretical justifications of school mathematical practices including specifications of the application conditions of school mathematical practices (discussing extreme and other exceptional cases) can show the new university praxeology without neglecting the school level one.

Aspect (a) can be more easily included in an Analysis I course than (b). Whereas (a) makes explicit some implicit socio-mathematical norms of a course. The inclusion of considerations of (b) would imply that the comparison of school mathematical thinking and reasoning and university mathematical reasoning becomes a topic of a university mathematics course itself.

In school teaching, the explicit discussion of errors, preconceptions and misconceptions of students is seen as relevant, when conceptual change is the aim. This can be considered as a challenge of university mathematics, too. At least, the above example shows that school mathematical practices seem to coexist in students' minds together with new university mathematical practices and can lead to poor results in examination.

Discussing these changes from school to university mathematics has become the topic of some (non-obligatory) pre-university bridging courses. It is also a topic in course for future teachers on the didactics of calculus at the school level, in order to support the second transition back to school but does this does not help with the first transition and discontinuity. However, I know of no Analysis 1 course that has taken up this challenge.

Supporting students in developing adequate concept definitions and concept images at university: The case of the convergence of sequences

In her dissertation which I supervised, Laura Ostsieker (2018) developed a non-compulsory 4-hour workshop for students of an Analysis 1 course that should support them in constructing the formal definition of convergence of a sequence in an attempt of guided reinvention. The workshop was situated some weeks after the start of the Analysis 1 course, just before the formal definition was introduced in the lecture. The study builds on work by Roh (Roh, 2010a, 2010b; Roh & Lee, 2017) and Oehrtman (Oehrtman, Swinyard, & Martin, 2014; Oehrtman, Swinyard, Martin, Roh, & Hart-Weber, 2011). The basic idea – which is taken from Przenioslo (2005) -- for setting up a learning environment is to give students a set of examples of sequences that are called to be "convergent" and others that are called non-convergent and ask the students to construct a definition of "convergence" where the examples are examples of, respectively non-examples (see initial task formulation in Figure 4). The set of examples was chosen to include examples that often have not become part of students' concept image after students have learned the formal definition. Examples include sequences that are alternating around their limit or where some (a finite or infinite number of) elements are equal to the limit itself. Figure 5 shows the set of examples and one non-example. Scatter plot graphs of the sequences $(n, a_n), \dots\ (n, x_n)$ were presented to the students, and the students were asked to relate the formal definition to the respective graph and to "formulate characteristic properties of every sequence". After that the task was

"The sequences $(a_n) \dots (f_n)$ are called convergent to the limit 1; the sequence (x_n) is not convergent. Describe the joint property as good as possible that the sequences $(a_n) \dots (f_n)$ have and the sequence (x_n) has not."

Figure 4: Initial task formulation (Ostsieker 2018, p. 86; my translation)

Laura Ostsieker had prepared a set of prompts for supporting the students when or if they got stuck. The workshop was offered twice. The second time, the revised conception of the workshop (tasks, visualisations, prompts) was offered to a different group of students one year later. The research study followed a design-based research framework. Hypothetical learning trajectories were developed for the first workshop, revised for the second one. Results from the retrospective analysis of the second version were provided for future implementation together with developing a local instruction theory. We cannot present Ostsieker with her detailed research questions and great results in this paper, but we will focus on one aspect: How did the formulation of the task change from the first to the third version? How does this change reflect some of the results of Laura Ostsieker's empirical study?

$$(a_n)_{n\in\mathbb{N}} = \left(\frac{n+1}{n}\right)_{n\in\mathbb{N}}$$

$$(b_n)_{n\in\mathbb{N}} \text{ mit } b_n = \begin{cases} 1-\frac{1}{n} & n \leq 125 \\ 1 & n > 125 \end{cases}$$

$$(c_n)_{n\in\mathbb{N}} \text{ mit } c_n = \begin{cases} -3 & 200000 \leq n \leq 500000 \\ 1 & \end{cases}$$

$$(d_n)_{n\in\mathbb{N}} \text{ mit } d_n = \begin{cases} 1 & \text{n Vielfaches von 10} \\ 1+\frac{1}{n} & sonst \end{cases}$$

$$(e_n)_{n\in\mathbb{N}} \text{ mit } e_n = 0,\underbrace{9\ldots9}_{n} \ \forall n \in \mathbb{N}$$

$$(f_n)_{n\in\mathbb{N}} \text{ mit } f_n = \begin{cases} 2 & n = 10 \\ 1+(-\frac{1}{2})^n & sonst \end{cases}$$

$$(x_n)_{n\in\mathbb{N}} \text{ mit } x_n = \begin{cases} 2 & \text{n Vielfaches von 10} \\ 1+\frac{1}{n} & sonst \end{cases}$$

Figure 5: The set of sequences (a_n) ... (f_n) are examples for the concept to be defined, (x_n) is a non-example; “Vielfaches” means multiples; “sonst” means “else” (Ostsieker 2018, pp. 84).

The task was challenging on two levels. We anticipated problems within the transition from an intuitive notion and verbally formulated attempt to characterize “joint properties” to a formal definition for the concept of convergence – given the very long history in mathematics that finally led to a formulation of the modern formal definition. What we did not anticipate were students' problems on a meta-level (see also Schüler-Meyer, 2018 for the role of the meta-level). They had to decide when a characterization they had developed with verbal and symbolic elements should be accepted as satisfactory. We had prepared the prompt "check whether your characterization is clear enough that it can be decided for every sequence whether it fulfils the characterization or not" to motivate and direct students for doing the next steps. In the second version of the task, we added this to the task itself to support a more self-regulated process of concept definition.

In retrospect, this was not surprising. Although the students had observed for several weeks how a mathematician was defining concepts and was using these definitions in theorem proving, students themselves had had no experience at all with activities of "defining concepts", which is typical when mathematics is presented like a ready-made product and not as a process.

Results of Ostsieker's empirical study include a new formulation of the initial task and further elaboration of the support measures for the different steps of the process. I will cite the final third formulation and use it as a starting point for describing some of her results. This third formulation is a result of the retrospective analysis, but it was not yet tested empirically in a new trial.

1. *"We are going to discover the definition (in university mathematics) of the convergence of a sequence to 1. The concept has to be defined in such a way that the sequences (a_n) ... (g_n) are convergent, and the sequence (x_n) is not.*
2. *Aim at formulating a joint property of the sequences (a_n) ... (g_n) that the sequence (x_n) does not possess as one single condition.*
3. *This condition has to be formulated in such a way that somebody to whom this formulation is presented can objectively decide and argue for every arbitrary sequence, whether this sequence has the property or not. Every person to whom this formulation is presented should come to the same conclusion about every specific sequence.*
4. *If you have the opinion that you have formulated an adequate property, check for all the example sequences whether they possess this property and check whether (x_n) does not have this property. If some of these checks are negative, revise your formulation." (Ostsieker 2018, p. 542, my translation. The numbering was added to make references in the following text clearer).*

Figure 6: final task formulation after two iterations of the workshop

The formulation in 1. explicitly asks for a mathematical definition and not only for a "property" so that it is possible to refer to socio-mathematical norms for definitions when students present formulations of a property that does not qualify for a definition. A "definition in university mathematics" was added because some students had entered the debate with a school-mathematical concept of "convergence" according to which some examples were not convergent that were claimed to be "convergent" by the task. This happened when students had a school-mathematical convergence concept that calls sequences convergent only when the elements are approaching a limit (monotonically) but never reaching it. We did not expect that some students did not accept the rules of our "game", so we choose to make it explicit that we have to distinguish by name possible previous concepts from the new concept that is to be defined. This reformulation may help; however, self-confident students maybe not willing to give up a concept that they found useful in the past. This observation may point to limitations of the approach Laura Ostsieker (2018) and Przenioslo (2005) have chosen. The approach does not provide problems that motivate a change of previous definitions because of theoretical or practical reasons but forces the students to accept that there is some hidden good reason in the choice of the provided examples. In the sense of Freudenthal's didactical phenomenology, no actual problem situation is presented that motivates the creation of a new concept (Freudenthal, 2002).

The formulation in 2 ("one single condition") reflects situations that occurred in the students' debates. Some groups characterized several subgroups of the examples adequately and then formulated the definition by several statements connected by "or". For instance, "convergence" can mean monotonically approaching a number with arbitrary closeness but not reaching it" or "being equal to this number from a certain *n* onwards" or "alternating around that number with …" or doing one of these things with a finite number of exceptions" etc. This solution even if it fulfils all criteria of rigour, would be considered as "inelegant" in mathematics. This phenomenon reveals another norm that mathematical definitions have to fulfil. Students in the first two workshop were right in claiming that this requirement was not specified in advance. That is why it was added.

The formulation 3 reacts to the situation that some students were satisfied with formulations for themselves. Pointing to potential “readers” of the definition should initiate a reflection of the preciseness and understandability of the definition and motivate students to check their definition with further self-created examples or use other students’ views. This extension of the task formulation may be of practical value. However, students may need to have more opportunities to reflect themselves on criteria that a mathematical definition must fulfil (Edwards & Ward, 2004; Ouvrier-Buffet, 2006; Zaslavsky & Shir, 2005). The formulation bears another problem. A mathematician would say that a mathematical definition must "in principle" allow to decide whether an object fulfils it or not. It will of course often happen that to decide whether a concrete object fulfils a definition or not may be considered as a severe mathematical problem to be solved in the future. Moreover, the formulation "every person" should come to the same conclusion is of limited practical: how can this be decided?

The formulation 4 reflects the following observation. In the anticipated learning trajectories, it was assumed that the students work on a successive improvement of their formulations of a definition, creating a first version of the definition from a limited set of examples, then systematically testing this formulation on all initial examples and also on new ones, then revising the definition and so on. Such a systematic approach was seldom observed. Therefore, the formulation was added to foster such an approach. The reason for the relatively unsystematic approach towards concept definition may not only be due to limited experiences in creating mathematical concept definitions, but it is plausible that students may have seldom met concept definition tasks or situations that require new concept definitions in their school and everyday life. In sum, these results also support the need to make implicit norms explicit and provide more extensive experiences for students to actively and reflectively participate in the new culture.

Concluding remarks

The transition from school Calculus to university Analysis is complex and multifaceted. Our examples pointed to problems that more explicit enculturation into mathematics at university level may help to overcome.

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