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TWG3 report: University Mathematics Didactic Research on Number Theory, Algebra, Discrete Mathematics, Logic

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INTRODUCTION

Ten oral communications were presented, reflecting the variety of the themes of the group: Number Theory, Algebra, Discrete Mathematics, Logic. In detail, two papers on Linear Algebra (Span, Linear transformation) were presented; 2 papers on Abstract Algebra (Group theory, the concept of ideal); 4 papers on Logic, Reasoning and Proof (syntax and semantic, Mathematical induction and recursion, Backward reasoning, personal meaning of proof); 2 papers on innovative teaching (first-year university students, Geometry capstone course). They were presented during sessions 1 and 3, being followed in each case by a discussion session nourished by the issues raised in the communications. Thirty-eight participants were registered for the sessions in this thematic working group. The number of attendees varied between 29 and 19, from Tunisia, Europe, North and South America and Japan; this was a challenge, due to the differences in local time zones.

SYNTHESIS OF THE COMMUNICATIONS

The two papers on Linear Algebra were presented respectively by Mitsuru Kawazoe (Japan) and Asuman Oktaç (Mexico). Mitsuri Kawazoe’s paper is entitled *Relation between understandings of linear algebra concepts in the embodied world and in the symbolic world*. In this study, linear (in)dependence and basis were focused on, and the relation between understandings of them in the embodied world and the symbolic world. I includes a study of the effectiveness of an instruction emphasizing geometric images of them. The main results of the study were the following: 1/ conceptual understanding of linear dependence of four spatial vectors such that any three of them do not lie on the same plane was positively associated with the understanding of the basis in the symbolic world. 2/ A geometrical instruction had not improved understanding of linear dependence of such vectors; indeed, in both pre-test and post-test, this task showed to be problematic for nearly half of the students.

Asuman Oktaç presented a paper written with by Diana Villabona, Gisela Camacho, Rita Vasquez and Osiel Ramirez on *Process conception of linear transformation from a functional perspective*. The paper discusses student conceptions involved in the construction of conceptions about a domain, image and inverse image of a linear transformation from IR^2 to IR^2 as well as the relations between these notions. The authors present the design of a set of tasks that allow exploring different facets of the above concepts, evidenced by the analysis of the production of a student. Thanks to the design of the instrument, it was possible to highlight some conceptions that may not be evident in typical teaching situations.

The two papers on Abstract Algebra were presented respectively by Koji Otaki (Japan) and Julie Candy (Switzerland and France). Koji Otaki presented a paper written with Hiroaki Hamanaka and Ryoto Hakamata entitled *Introducing group theory with its raison d'être for students*. This paper reports results of a sequence of didactic situations for teaching fundamental concepts in group theory, e.g., symmetric group, generator, subgroup, and co-set decomposition. Students in a pre-service teacher-training course dealt with such concepts, together with card-puzzle problems the analysis of which provide students with the raisons d'être of these concepts.

Julie Candy presented a paper entitled *Etude de l'enseignement du concept d'idéal dans les p*remières années postsecondaires: élaboration de modèles praxéologiques de référence. The paper presents the construction and interpretation of a praxeological reference model for teaching the concept of ideal in the first two post-secondary years in France, in two different institutions, before this concept is taught systematically in Ring Theory. The model allows a comparison of the choices made by the two institutions and a first discussion of the implementation of structuralist thinking, in the perspective of the teaching of abstract algebra in the third year of university.

The four papers on Logic, Reasoning and Proof were presented respectively by Zoé Mesnil (France), Nicolas Leon (France), Ines Gómez-Chacón (Spain) and Sandra Krämer (Germany). Zoé Mesnil presented a paper written with Virginie Deloustal-Jorrand, Michèle Gandit, and Mickael Da Ronch, entitled *Utilisation de l'articulation entre les points de vue syntaxique et sémantique dans l'analyse d'un cours sur le raisonnement*.The authors highlight the relevance of the articulation between syntax and semantics in proof and proving activities. With this lens, they present a logical and didactical analysis of a university course entitled "Mathematical Reasoning", relying on interviews with teachers, worksheets and an assessment test. The case study presented here is the first step for a comparative study aiming at characterizing the teachers' views on proof and proving, as a preliminary before studying students' appropriation of the various aspects of proof and proving.

Nicolas Leon presented a paper, written with Simon Modeste and Viviane Durand-Guerrier, entitled *Récurrence et récursivité: analyses de preuves de chercheurs dans une perspective didactique à l'interface mathématiques*. The authors present the analysis of researchers' proofs of the equivalence of two definitions of the concept of tree in graph theory, one of the two definitions being recursive and the other not. The analysis aims to shed light on the relationship between the notions of recurrence and recursion, as perceived by experts. The authors will rely on the results of this study when designing didactic sequences aiming to work with students on recurrence and recursion and their interactions.

Ines Gómez-Chacón presented a paper written with Marta Barbero and Ferdinando Arzarello entitled *Backward reasoning and epistemic actions in discovery processes of strategic games problems*. The authors focus on the epistemic and cognitive characterization of backward reasoning in strategy games problems with PhD students in a Spanish and an Italian university. They report a case study showing the process of discovery that a PhD student carries out to formulate a general recursive formula. They propose a unified framework that allows focusing on both short-term and long-term processes in students' activities. Sandra Krämer presented a paper written with Leander Kempen and Rolf Biehler entitled *Investigating high school graduates' personal meaning of the notion of "mathematical proof*". In this paper, the authors report on the results of a pilot study to investigate high-school graduates' personal meaning of mathematical proof. By using proof tasks and a following interview phase with meta-cognitive questions, they describe students' personal meaning of the notion of mathematical proof and show, among others, that some students hold different meanings of the word "proof" simultaneously.

Each of the last two papers presents innovative courses in teaching mathematics. They were presented respectively by Patrick Gibel (France) and Max Hoffmann (Germany). Patrick Gibel presented a paper written with Isabelle Bloch entitled *Analyse des effets d'un dispositif innovant sur l'évolution des représentations des étudiants en première année de licence de mathématiques*. The authors present an innovative course set up at the University of Pau in order to help undergraduate students to overcome difficulties in the secondary-tertiary transition. A main mean is to involve students in research into mathematical problems. An example situation is described and analysed.

Max Hoffmann presented a paper written with Rolf Biehler entitled *Designing a Geometry Capstone Course for Student Teachers: Bridging the gap between academic mathematics and school mathematics in the case of congruence*. The authors present a geometry course for upper secondary student teachers aiming to show links between academic mathematics and school mathematics. In the paper, they focus on the concept of congruence, illustrating how specific aspects of the course are used to systematize the mathematical background of the topic, thus enabling future mathematics teachers to diagnose and react in fictitious teaching situations professionally based on subject matter knowledge in mathematics. Finally, they provide examples of learning activities in the course and first results of analysing students' work.

The paper by Khalid Bouhjar, Christine Andrews-Larson, and Muhammed Haider, *On students' reasoning about span in the context of Inquiry-Oriented Instruction*, has not been presented, but is available in the proceedings. The authors analyse differences in reasoning about span by comparing the written work of 126 linear algebra students who learned through a particular inquiry-oriented (IO) instructional approach compared to 129 students whose instructors used other instructional approaches. Their analysis of students' responses to open-ended questions indicated that IO students' concept images of the span were more aligned with the corresponding concept definition than the concept images of non-IO students. Additionally, IO students exhibited richer conceptual understanding and greater use of deductive reasoning than Non-IO students.

MAIN ISSUES DISCUSSED DURING THE SESSIONS

The main theoretical and methodological issues discussed were 1/ the role of ATD (Anthropological Theory of the Didactic) for analysing, designing, giving access to the raisons d'être of a mathematical topic; 2/ the means to address the complexity of mathematical notions with students (e.g. Cayley diagram in group theory, recursion, strategic games in 3D); 3/ the relevance of analysing data through the lens of concept images versus concept definition, and of considering the impact of the choice of definition in students' activities (e.g. definition of the image starting from the domain or codomain; 4/ what can we infer from case studies depending on the two following cases: 4.1: a significant amount of data have been analysed – a representative case; 4.2 a small number of interviewees, but a diversity of profile providing a great richness in the data. In both cases, it is not possible to generalize, but such a case study might contribute to enrich a priori analysis and identify candidates for operational invariants. Issues on proof and reasoning prevailed in the four papers focusing on this topic, but also in other papers, and were widely discussed. Several questions on proof classification were raised: what counts as an empirical argument? What is the difference between generic proof, narrative proof, symbolic proof? What links exist with the classification of the type of proofs by Balacheff? How to distinguish between correct and incorrect proof, considering the audience of a proof? Some participants wonder if there is a consensus among university teachers on what is a mathematical proof; more precisely, in a didactical transposition perspective, is there a common reference on proof that would make easier its teaching and learning. The answer is that this is not obvious because there might be dependence on the educational context or personal views of teachers. In some cases, a local consensus may exist among a pedagogical team.

Different and related (necessary) aspects of teaching proof have been considered in the discussions: 1/ showing proofs to students seems necessary but is clearly not sufficient; 2/ solving problems with not too obvious solution to motivate the need for proof; 3/ teaching what is a proof and its role in mathematics to provide students with meta-knowledge on proof in mathematics. 4/ having students experience how to construct and analyse proofs, in their mathematical and logical dimensions; 5/ considering not only proof but also proving as a practice; 5/ teaching proof as a separate topic or integrated into teaching mathematical topics (with reflections on proof)? 6/ considering the role of proof on conceptualization and the reverse.

FURTHER RESEARCH AVENUES

Finally, we have identified main open questions and research areas deserving more attention for the years to come. A promising avenue of research is addressing the second transition of Klein (in programs for mathematics teacher education), by developing innovative teaching modules to allow students in a teacher training program to deeply understand the relationships between university mathematics and school mathematics in a professional perspective. There are convincing examples but also several challenges: 1/ finding relevant topics with strong epistemological foundation (e.g. congruence, symmetry, integration, proof); 2/ developing collaborations between university teachers and researchers in didactics of mathematics (some might be both) for implementation and analysis; 3/ managing to implement it, depending on the context: department of mathematics *versus* faculty of education; 4/ finding a way of dissemination of research results toward mathematics university teachers. Exploration of paths of collaboration between mathematicians and researchers in mathematics education, considering various institutional contexts: 1/ having researchers in didactics of mathematics in a department of mathematics; 2/ having professional mathematicians in a faculty of education; 3/ developing collaboration in a doctoral programme - co-supervision of PhD students; 4/ designing training modules for mathematics university teachers (mandatory in many countries – should also be specific to the domain of mathematics, not just general pedagogy); 5/ organizing workshops aiming at participants to get acquainted with didactic aspects of the teaching and learning of university mathematics.

Address proof and proving issues at all level of university mathematics. 1/ going on investigating the possibility of a common background (*versus* specificity) on proof and proving for developing university mathematics (both undergraduate and graduate) studies; 2/ deepening studies on the role of proof and proving in conceptualization on advanced topics (e.g. number theory, linear and abstract algebra, algebraic topology, algorithms, discrete mathematics, recursion); 3/ developing research on proving as a practice linked to solving problem: epistemological and didactical issues; 4/ developing students' meta-knowledge on proof as a topic of its own on top of students' experiences of proof and proving as part of problem-solving processes; 5/ working in a given axiomatic *versus* exemplary participation in an axiomatization of a domain; going on addressing logical issues in mathematics and establishing links with mathematics and computer science. These issues are in line with than some of those identified and discussed in Chellougui et al. (2021).

REFERENCES

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