In: Hausberger, T., Bosch, M., & Chelloughi, F. (Eds.). (2020). Proceedings of the Third Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2020, 12-19 September 2020). Bizerte, Tunisia: University of Carthage and INDRUM.

Designing a Geometry Capstone Course for Student Teachers: Bridging the gap between academic mathematics and school mathematics in the case of congruence

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At Paderborn University, a new 6th semester geometry-course for upper secondary student teachers has been designed and taught by the first author of this paper. To show links between academic mathematics and school mathematics we established so-called interface weeks. These are weeks during a course in which lecture, exercises and homework focus on topics that are related to the normal canon of content but specially chosen for their relevance in school contexts. In this article, we want to present our design for an interface week on the topic of congruence. In order to do so, we first illustrate how so-called interface aspects are used to systematize the mathematical background of the topic, thus giving future mathematics teachers the chance to act professionally. We then show examples of learning activities and first results of the accompanying research.

Keywords: Transition to and across university mathematics, Teaching and learning of specific topics in university mathematics, Geometry, Student Teacher, Capstone

INTRODUCTION

In his well-known quote, Klein (1908, p. 1) describes two discontinuities, which must be clearly distinguished. The *first discontinuity* is a perceived disconnectedness between school mathematics and the academic mathematics that students encounter when they enter their university studies. Klein focusses exclusively on the aspect of mathematical knowledge. In current transition research, also differences at the level of teaching/learning methods or social challenges inherent in the transition are seen as responsible factors for the difficulties which students experience when they start their studies. These difficulties can even cause some students to completely drop the study of mathematics. Interventions that specifically address the transition problems at the aspect of mathematical knowledge explicitly build on existing knowledge and previous mathematical experience from school when designing teaching/learning processes in university mathematics. The focus of these efforts is the acquisition of competences in university mathematics. Although was formulated by Klein in the context of teacher education, the first discontinuity is also relevant for students who aim to major in mathematics. The *second discontinuity* assumes that students have acquired knowledge in university mathematics. However, students often perceive this mathematical knowledge as not very relevant for their future professional work. Solutions therefore aim, to identify the contents of academic mathematics that can be connected (in the sense of supporting professional teaching) to school mathematics and in a second step to develop learning opportunities that support students in discovering connections. The aim is to better enable students to use their background in university mathematics as a basis for professional acting as a teacher. In most (German) universities, such linkage is provided in additional courses on the didactics of mathematics. Our new approach is to enhance the mathematics course itself so that the specific course on the didactics of geometry can focus on pedagogical content knowledge. The mathematics course is to be enriched by learning opportunities that help students to take a mathematical perspective on a profession-oriented situation (e.g. reacting to a student's contribution or analysing a textbook page) and to act with the necessary professional knowledge.

THEORETICAL BACKGROUND

Talking about a discontinuity between school mathematics and academic mathematics requires analysing differences between the two. Dreher, Lindmeier, Heinze and Niemand (2018) summarize how these differences have already been described by Klein and, more recently, by Wu (2011) and other authors:

> Mathematics as the scientific discipline taught at university has an axiomatic-deductive structure and focuses on the rigorous establishment of theory in terms of definitions, theorems, and proofs. It usually deals with objects that are not bound to reality […]. [M]athematical objects [as taught at schools] are often introduced in an empirical manner and bound to a certain context. Concept formation […] is […] often done in an inductive way […]. Mostly intuitive and context-related reasoning is more in the focus than rigorous proofs. (Dreher et al., 2018, p. 323)

In order to link these two types of mathematics, we follow the idea of *mathematical background theories* (e.g. Vollrath, 1979). Topics of school mathematics are characterized by several locally ordered domains, which are mostly unconnected to each other. They are often built up from an empirical phenomenon and are at the end networks of terms and concepts that are logical in themselves. (Freudenthal, 1973). Background theories phrased in the language of academic mathematics can now contribute to the foundation of these locally ordered domains in two ways: On the one hand, they can help to connect the domains with each other and thus clarify conceptual relationships and bring statements from the different areas into logical connections. On the other hand, such background theories provide the basis for the local ordering and selection of content foci within the individual domains. Based on extensive research, Ball and Bass (2002, p. 11) give a list of typical mathematical job tasks that a mathematics teacher has to master in everyday teaching. This list has been extended by Prediger (2013). Four of these job tasks that are important for our project are the following:

A teacher must …

- be able to master requirements set for students by her- or himself at different levels
- analyse and evaluate approaches (used in e.g. textbooks)
- analyse and rate student contributions and react to them in a way which is conducive to learning

- analyse mistakes of students and react in a way which is conducive to learning

(Prediger, 2013, p. 156)

The identification of the corresponding mathematical background theory is a prerequisite for professional teaching, since it enables a teacher to correctly analyse a given situation. With sufficient background knowledge, a mathematical perspective on a typical professional problem can be taken and a solution can be worked out in the context of this perspective. In the last step, this solution then must be didactically transferred back into school mathematics and adapted to the mathematical horizon of the respective school students. At this point, we would like to emphasize that this is just one of several perspectives that can be taken on typical professional situations, but it is the one in which the mathematical background plays the most important role, which makes it relevant for the design of lectures in mathematics. In terms of teachers' professional knowledge, our aim is to establish links between school mathematics and academic mathematics in the sense of school-related content knowledge (SRCK) (Dreher et al., 2018). This construct supplements the known facets of content knowledge and pedagogical content knowledge with a profession-specific component. The latter consists of three facets, namely: Knowledge about the school curriculum and its structure as well as the understanding of its mathematical legitimation, secondly the knowledge of the interrelations between school mathematics and academic mathematics both top-down, and thirdly bottom-up. We call learning opportunities that evoke the conscious passing through the described three-step process (Take the mathematical perspective. - Solve your problem. – Didactically transfer your solution) *interface learning opportunities (to bridge the second discontinuity)* and thus generalize the term *interface task* as used for example by Bauer (2013). Our research interest lies in the development and evaluation of interface learning opportunities with the aim of identifying generalizable principles for success and failure and formulating general design principles for interface activities.

ABOUT THE GEOMETRY COURSE

The course in which our project takes place is located in the 6th semester of the degree programme for future upper secondary math teachers. Since it was newly introduced as part of a change in the study regulations, we had many design options for implementing the requirements of the module manual, which are: an axiomatic system for Euclidean geometry should be treated according to the module description of the course and, the role of the parallel postulate should be discussed using a model of hyperbolic geometry. For this purpose we use an axiom system developed by Iversen (1992). His system is equivalent to the known axioms of Hilbert, but it needs fewer axioms, which are of course more charged with content (as for example, the axioms for $\mathbb{R}$ are already included.), but also more intuitively understandable. In addition, we also consider the Euclidean plane by means of analytical geometry using Euclidean motions and thus include the geometry as it is usually treated in upper secondary school. In this way we can take different views on typical terms of school geometry (straight lines, orthogonality, reflections, etc.). The course took place for the first time in the summer semester 2019 with about 25 active participants. The presence part consisted of a weekly two-hour lecture and two two-hour tutorial groups of about 13 students each. As usual in mathematics lectures in Germany, weekly homework assignments were set for the students to work on. Within this context, students also get tasks for linking school and university mathematics, which are part of a semester-accompanying so-called *interface ePortfolio*. We follow Bruder, Scholz and Menhard (2012) with this concept of an accompanying e-portfolio. It must be emphasised that the course differs significantly from the usual mathematics courses in university teacher education, which are usually the standard bachelor-of-science courses and thus do not specifically address the specific needs of student teachers. Most of the existing projects in which special courses for student teachers are developed are placed at the start of university education**.** The topic of elementary geometry is well suited for an exclusive teacher education lecture, because although it is an important topic for student teachers, it usually does not play a role in the subject studies.

RESEARCH DESIGN

We develop and study our interface activities within the framework of a design research approach following the methodology of Prediger et al. (2012). In our project we go through the following cycle three: (Step 1) *specifying and structuring the interface topic*, (Step 2) *(re)designing interface learning activities*, (Step 3) *use and research interface activities* and (Step 4) *developing and refining (local) theories*. The initial run just took place in summer semester 2019: In order to *specify and structure the interface topic (step 1)*, we must link a school mathematics topic with corresponding academic mathematics. The challenge now is to systematize the background theory so that it can be used as a basis for professional teaching. It is utopian to assume that teachers will later think, “Oh yes, lemma 4.2 can help me here.” Our approach is to work out so-called *interface aspects* that channel the work with the mathematical background in a typical professional situation. In the next section this will be explained by an example. We decided to focus our project on the interface topics *symmetry* and *congruence*. For *design* and later *redesign (step 2)*, we had to develop three types of interface learning activities: The lecture itself, tasks for the weekly tutorial groups and tasks for the weekly homework. In the first run, we were primarily dependent on experience from other mathematics lectures for student teachers, other projects on interface tasks and our conceptual considerations, which are described above. In the next two runs, we will also be able to build on our research results according to the principles of design research. For the *research on our interface learning activities (step 3)*, we have collected a lot of data both at the level of the cohort as a whole and in case studies. Details about the data collection will be presented in the next section. Based on the research results from step 3, we can then *develop and refine local teaching-learning theories (step 4)* that relate to the developed activities belonging to the selected interface topic. In this article we would like to illustrate the implementation of the different steps of the cycle using various examples from the *interface week "congruence"*.

DATA COLLECTION

The students' solutions of the interface tasks were scanned and will be evaluated with methods of qualitative content analysis. A small sub-group from each tutorial group was videographed in an extra room while working on the tasks. By evaluating the students' discussions, we hope to gain insights into the conception of interface tasks as well as into how students use the newly acquired background theory in their communication with peers. Some students were interviewed about their experiences during their work on the homework and their view on the tasks. This enables us to gain insights into the subjective perception of the learning processes during the interface week. In the following week, we used an acceptance questionnaire in the whole group with mainly closed items, which gives us a general picture of the perceived difficulty, comprehensibility, motivation, etc. of the interface tasks.

DESIGN OF THE INTERFACE WEEK ON "CONGRUENCE"

For one of the interface weeks, we chose congruence, which is an important topic of geometry teaching in lower secondary school. Among other things, the topic forms an important background for constructions with compasses and rulers and represents an essential method of geometric argumentation in school mathematics.

Mathematical Background

The first part of the lecture on the *interface week on "congruence"* deals with the clarification of the underlying academic mathematics.

We call a figure (that means subsets of $\mathbb{R}^2$) F congruent to G, if there is an (bijective) isometry $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ with $\varphi(F) = G$. We prove that congruence is an equivalence relation. That leads us to the question of what information is needed to unambiguously determine the equivalence class of a plane figure. Prominent tools for this are the congruence theorems for triangles, which are also proved in the lecture.

Congruence in school mathematics

Weigand et al. (2018, p. 202) describe *congruence* as an important basic concept for topics of lower secondary geometry. This includes constructions with compasses and rulers, justifications and proofs as well as the determination of lengths and area contents via congruent subfigures. Congruence can be introduced as a basic concept explained on the enactive level (in the sense of fitting when laying one figure on top of the other) or based on the theory of Euclidean isometries (Weigand et al., 2018, p. 203).

Interface Aspects

In accordance with step 1 of our research design, we worked out the following four *interface aspects* for *congruence* as a kind of meta-knowledge for constituting a systematized view of the mathematical background. The notion of "interface aspect" is an important theoretical concept of our approach.

1. *Aspect of quantities with identical size:* Because there is always an isometry (by definition) between a figure and a congruent figure, it is guaranteed that congruent figures will match in several geometric quantities. This applies especially not only to the border of the figure and the distance between corner points, but also to the dimensions of other objects that can be constructed from the figure (diagonals, intersections, incircles, …) and their equivalent objects in the congruent figure.
2. *Aspect of relation:* Congruence is an equivalence relation on the power set of $\mathbb{R}$. The statement delivers characteristics that are connected intuitively with the concept of congruence: Each figure is congruent to itself (reflexivity). If A is congruent to B, then B is also congruent to A (symmetry). It is only in this way that the phrase "figures are congruent *to each other*" is meaningful. If two figures are congruent to a third one, they are also congruent to each other (transitivity).
3. *Aspect of classification:* The aspect of relation provides a disjunctive classification of all subsets of $\mathbb{R}^2$ into congruence classes. The classification aspect now emphasizes the typical question of identifying and describing particularly relevant congruence classes, as well as working out common properties of all figures within these classes. The latter is a specification in the sense of the aspect of quantities with identical size. The following question about the smallest possible amount of information for the unambiguous assignment of a figure to its congruence class leads to the classical congruence theorems.
4. *Aspect of mapping:* While the aspect of quantities with identical size statically compares the measurable properties of congruent figures, the question as to how one figure can be "transformed" into the other is part of the mapping aspect: (1) For every two congruent figures there exists by definition a mapping (bijective isometry) which transfers the figure into the other. (2) We can always express these mappings by the composition of a maximum of three straight line reflections (three-reflections theorem). (3) This mapping is always a glide reflection, a rotation or a translation. All this is especially valuable if you have proven the congruence of two figures over a congruence theorem. Automatically we already know about the existence of such a mapping.

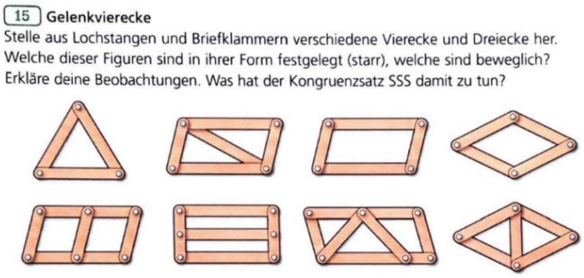
We can now use the *interface aspects* to illustrate links between school mathematics and academic mathematics. Many activities in dealing with congruence in school use congruence in the sense of the *aspect of classification*. This includes in particular constructions: Unambiguous constructability of triangles or other figures means that all figures that can be constructed from a given set of sizes are in the same congruence class. Also, the solving of plane triangles and the associated question of a minimum number of determining characteristics for a congruence class of triangles fall under this aspect. The determination of lengths or area contents using (sub)figures is based on the view of the concept of congruence described in the *aspect of quantities with identical size*. Proofs can use congruence in the sense of a mixture of both aspects: Congruence theorems are used to identify congruent figures with the aim of connecting different quantities with identical sizes. The *aspect of relation* is contained in the usual wording in textbooks: "congruent to *each other*". This is well-defined, only because congruence is a symmetrical relation. And finally, when the enactive action of laying congruent figures on each other, through reflections etc., is formalized, the *aspect of mapping* always comes into play.

The interface aspects take up all areas of the mathematical background of the topic “congruence” and systematize them on a level independent of the degree of abstraction. Our hope is that the aspects are suitable to support the taking of a meaningful mathematical perspective in a typical professional situation. We have shown that for each aspect there are areas of school mathematics in which it can be addressed.

Examples for interface tasks on the topic of congruence

The following task was used in the exercise group on congruence.

Consider the following textbook task *(Neue Wege 7, NRW (2014), p. 195)*:



a) For each of the figures shown, consider whether their shape can be changed or not. Explain your observations and discuss which role the theorem SSS plays.

b) Discuss, the role of the interface-aspects of *congruence* in the given task.

c) Based on your considerations in a), consider at least one congruence theorem for squares in your group and justify its validity. (our translation).

The professional relevance of the task lies in the domains (in the sense of Prediger, 2013) "be able to master requirements set for students by her- or himself at different levels " and "analyse and evaluate approaches (used in e.g. textbooks)". The content-related focus of the task lies in the *aspect of classification*, since it is a question of a sufficient number of given quantities for the unambiguous description of congruence classes (of quadrangles). The *aspect of quantities with identical size* also plays a role, as the modifiable figures are characterised by the fact that there are lengths (e.g. of diagonals) which are not clearly defined by the given lengths.

Quotes from students working on the task

As already described, we have video recordings from the tutorial meetings at our disposal, which we evaluate using methods of qualitative content analysis (Kuckartz, 2018). This analysis is still work-in-progress. One of our foci is students’ use of the interface aspects that the students have learned in the previous lecture. We now want to illustrate three of our categories using examples from the discussion of a group of three students on the above task (Table 1).

Category name	Category description	Example
proper and expected use of an interface aspect	The students use an interface aspect in an appropriate manner, in accordance with our preliminary considerations	(referring to the aspect of quantities with identical size) “[…] If we look at the [undrawn] diagonals, the lengths change if they (the figures) can move.”
proper but unexpected use of an interface aspect	The students adequately address an interface aspect in a way not anticipated by us.	Student uses the aspect of mapping in the context of the moving figures as follows: In the case of a moving figure, the different possibilities do not go into each other through reflection, rotation or translation and are therefore not congruent.
Incorrect use of an interface aspect	The students use an interface aspect in the wrong way.	(referring to the aspect of classification) *“[…] Then we have in principle two classes, one where something can move and one where nothing can move. […]”*

Table 1: Categories of content analysis of video recordings of tutorial groups.

Especially the second two categories (proper but unexpected use and incorrect use) are very valuable for redesigning the learning activities according to our Design Research cycle. Passages which fall in the second category expand our knowledge about anticipated learning processes and thus indirectly also about possible problems. The example of the third category described in the table provides a reason to reconsider the formulation of the *aspect of classification*. The aim of the redesign is to make it clear to the students during the course that the classes are congruence classes and not sets defined by other features such as "All figures that are created by deformation of a moving figure". The first category shows us where the aspects seem to function exactly in the intended sense.

FURTHER SELECTED RESEARCH RESULTS

As described above, we also studied the interface week with a questionnaire with closed items. At this point, we like to present first results of two scales that deal with the self-assessment of students after the interface week (Table 2). Figure 1 shows a histogram of both scales. It becomes clear that the maj-ority of the students in both scales assess themselves rather positively and - after the interface week - are confident that they can work with the background theory in the field of congruence and act professionally on this basis. The interesting question is what role the interface aspects play here. The item 'The interface aspects to congruence have helped me to better structure the mathematical background of the topic congruence.' was answered by all 21 surveyed students on a Likert scale from 1 to 5 with 4 (rather true, 14 students) or 5 (completely true, 7 students).

Scale	Description	Example Item	scale consistency
Background theory self-perception (5 items)	Self-perception of the ability to act in the background theory of congruence.	*I think that after the interface week, I can precisely define school mathematical terms from the area of congruence.*	$\alpha = .85$
Professional acting self-efficacy expectation (9 items)	Expectation to self-efficacy to act professionally in the area of congruence as a teacher (Item formulations are based on the job-tasks described by Prediger (2013)	*I think that after the interface week, I can analyse and evaluate a textbook excerpt on the topic of congruence.*	$\alpha = .79$

Table 2: Two scales of the acceptance questionnaire to the interface week congruence.

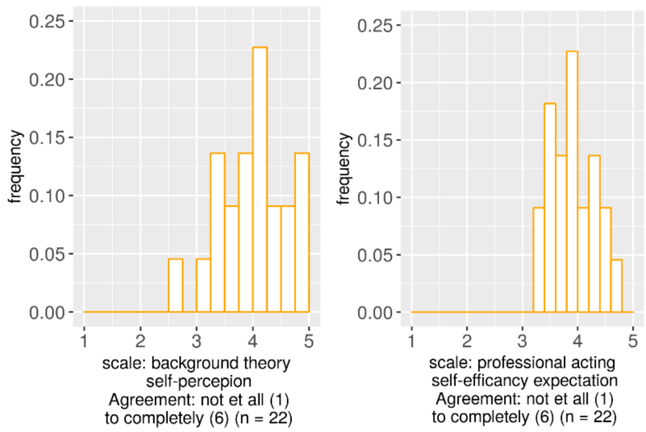


Figure 1: Histograms of the two scales from Table 2.

This is first and foremost a positive assessment, but it also shows that there is still room for improvement. Here, the results of the video evaluation described above, and the analysis of homework provide a good basis for a reformulation of the aspects in order to increase their accessibility of. The fact that the aspects are perceived as helpful is also supported by first results of our interview study, as the following original quote of one of the interviewed students shows:

> [...] The aspects already were very useful. So especially the whole thing of getting a structure like that in there. That you say: I have the aspects and can then relate them to them [to situations] and could think about which aspect I could use where or emphasize where.

FURTHER PERSPECTIVES

In the next step, we will continue the descriptive evaluation of the collected data and connect them with each other in order to be able to make more profound statements about learning processes in the interface weeks. However, it is already clear that the idea of interface aspects is positively received by the students, but it is still unclear, how such aspects can be schematically set up for other mathematical concepts.

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