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Investigating high school graduates' personal meaning of the notion of "mathematical proof"

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In this paper, we report on the results of a pilot study to investigate high-school graduates' personal meaning of mathematical proof. By using proof tasks and a following interview phase with meta-cognitive questions, we will describe students' personal meaning of the notion of mathematical proof and show that some students hold different meanings of the word "proof" simultaneously.

Keywords: reasoning, proof, generic proof, personal meaning, example.

INTRODUCTION

Mathematical proof can be considered being a major hurdle for mathematics freshmen (Selden 2012, p. 293). However, when trying to teach mathematical proof and proving to first-year students, their previous knowledge on the topic has to be taken into account (ibid., p. 414). Besides learners' competencies concerning proof construction, reading, and evaluation, the personal meaning they assign to the notion of mathematical proof seems to play a crucial role in their mathematical behaviour (Harel & Sowder 1989). While different studies have investigated university students' proof competencies, we focus on students' knowledge concerning mathematical proof after graduating from high school. Accordingly, we aim at investigating students' personal meaning of proof as a part of their existing knowledge of mathematical proof when entering university. For this purpose, we rely on the study of Recio and Godino (2001). These authors elaborated on different personal meanings of mathematical proof. We expand their investigation to today's high-school graduates and deepen their approach making use of qualitative methods. When clarifying high-school graduates' personal meaning of mathematical proof, the socialisation process concerning proof in school mathematics can be elaborated. Moreover, the consensus on the meaning of mathematical proof has to be considered as an inevitable requirement for teaching mathematical proof at university. In this sense, it might become possible to link university studies to previous experiences from school mathematics and the enculturation process to higher mathematics can be undertaken more consciously. Finally, it might get possible to conceptualize or to expose popular misconceptions concerning mathematical proof. In this paper, we report on the design and the results of a pilot study, where four high-school graduates were asked to prove two mathematical claims and to explain and to validate their performances afterwards.

THEORETICAL BACKGROUND

Empirical findings from the literature

Kempen and Biehler (2019) evaluated first-year pre-service teachers' proof validation. In their study, 29.7% of the 37 first-year students rated a purely empirical verification as "correct proof" when starting their university studies. Selden (2012, p. 398 ff.) summarizes several problems of first-year students with mathematical proof and highlights i. a. a nonstandard view of proof (e.g., what constitutes a proof and how the proof process is interpreted). Following Kempen and Biehler (2019, p. 246 ff.), first-year students mainly link the concept of proof with some prototypes of proof, like the proof of Thales' theorem or the proof of the binomial formulas. Also, beginning students do not have much experience with proof construction. While proving at university is associated with the use of definitions, the application of theorems about abstract concepts and deductive reasoning (Selden & Selden 2007), the named examples from school mathematics display another 'concept' of proof: In school geometry, proofs make use of a figure to perform reasoning. In elementary arithmetic, many proofs utilize simple calculations using variables (e.g. in the proof of the binomial formulas). Besides, following the TIMS-Study in 1998, German high-school students showed only low abilities concerning the construction and evaluation of mathematical proof (compare Reid & Knipping, 2010, p. 68).

Categorization of students' proof productions

Recio and Godino (2001) investigated the proof competencies of first-year university students in Spain. In their study, i. a. 429 students entering university were asked to work on two elementary proving tasks. The authors conclude that the percentage of students giving a substantially correct mathematical proof to each problem is less than 50%. Only 32.9% of the students gave correct answers to both proving tasks. Interestingly, about half of the students (53.8%) formulated a purely empirically based answer to at least one of the given tasks. The authors classified student's proof attempts using the following set of categories: (1) "The answer is very deficient (confused, incoherent)", (2) "The student checks the proposition with examples, without serious mistakes", (3) "The student checks the proposition with examples, and asserts its general validity", (4) "The student justifies the validity of the proposition, by using other well-known theorems or propositions, by means of partially correct procedures", and (5) "The student gives a substantially correct proof, which includes an appropriate symbolization". Finally, the authors tried to link the named categories with personal proof schemes in reference to Harel and Sowder (1998). Answers of type (2), the mere empirical confirmation of a proposition, are connected with the "explanatory argumentative scheme", because "There is neither a true intention to validate the proposition, nor an intention to affirm the validity of the proposition for all possible cases." (ibid.). Answers of type (3) are considered to be in line with the "empirical-inductive proof scheme"; these answers are based on verifying the proposition by using particular examples, without the intention of justifying the general validity. In contrast to the former categories, answers of type (4) and (5) include the intention of verifying the general validity of the proposition by using deductive reasoning. Accordingly, answers of type (4) are connected with the so-called “informal deductive proof scheme”. Finally, the answers of type (5) follow a more formal approach, making use of a symbolic and algebraic language. These answers are assigned to the “formal deductive proof scheme”. When students’ performances in the two tasks seemed to correspond to each other, the authors interpreted the proposed categories as personal schemes of mathematical proof to describe students’ personal meaning.

Reid and Knipping (2010, p. 130 ff.) distinguish four kinds of proof or argument according to the representations involved. (1) “Empirical”: Those arguments in which specific examples are used but do not represent a general case, (2) “Generic”: Those arguments in which specific examples are used to highlight a general idea, (3) “Symbolic”: Those arguments in which words and symbols are used as representations, and (4) “Formal”: Those arguments in which symbols are used without semantic reference. In this paper, we make use of this categorization of proofs to categorize students’ proof productions. Since we are dealing with high-school students’ proof attempts, the fourth kind of proof will not appear in the analysis. We split the third type of proof to distinguish an increased used of words (“narrative proof”) and an increased application of symbols and variables (“symbolic proof”). An example of each type of argument is given below.

Research Questions

Based on the theoretical considerations above, we formulate the following two research questions: (1) How do upper secondary school students prove claims from elementary arithmetic and geometry? (2) Which personal meaning of the notion of mathematical proof can be assigned to the students?

METHODOLOGY

Interview design

In our study, four students from upper secondary school were supposed to work on two proving tasks, one from elementary arithmetic and one from elementary geometry (see below). Afterwards, the students were asked to explain their solutions and to answer metacognitive questions. We used a combination of task-based interviews according to Goldin (2000) and the Precursor-Action-Result-Interpretation (PARI) method (Hall et al., 1995) modified by Kortemeyer and Biehler (2017) for the use within mathematics education research. Both methods intend two main stages: the solving of mathematical problems and a following interview concerning the participants’ approach, their reasons for choosing it and the interpretation of their results. Mainly, the modified PARI methodology was added to develop and organise the interview questions in three phases (working individually on the task, recapitulating one’s solution with the interviewer, reflecting on one’s strategies and decisions).

In accord with Goldin (2000, p. 522 f.), the study was split into the following stages: (i) posing the questions with sufficient time for working on the tasks, (ii) minimal heuristic suggestions and assistance, if the participants display serious problems (e.g., "Can you tell me what the claim is about?"), (iii) questions concerning students' approaches (e.g., "What did you do?"), and (iv) metacognitive questions. In this last part, the students were questioned i. a. about their satisfaction with their solutions and the reasons for choosing the respective approach. We also asked the participants to evaluate their solutions in order to see if they consider them as correct proofs.

Task analysis and expected solution

The proving tasks used in the study should be accessible to all students and allow for different approaches. We followed the idea of Recio and Godino (2001) to use one claim from elementary arithmetic and one from geometry. This way, we also wanted to investigate if students' proving approaches and personal meaning on the notion of mathematical proof differ with respect to the subject. We replaced the first claim from Recio and Godino (2001) with a proving task from Biehler and Kempen (2013), because this task seemed to be more suitable for us. The second claim was taken from the original study. The named tasks were slightly modified for the use within this study. Finally, the participants were supposed to solve the following tasks:

<u>Task 1:</u> *Prove that the sum of an odd natural number and its double is always odd.*
<u>Task 2:</u> *We consider two adjacent angles α and β. Prove that the bisectors of α and β always form a right angle.*

In Table 1, a non-exhaustive set of expected solutions with regard to the categorisation of arguments according to Reid and Knipping (see above) is presented.

	Task 1	**Task 2**
empirical argument	$1 + 2 \cdot 1 = 3$ $3 + 2 \cdot 3 = 9$	$120° + 60° = 180°$ $60° + 30° = 90°$
generic proof	$1 + 2 \cdot 1 = 3 \cdot 1 = 3; 3 + 2 \cdot 3 = 3 \cdot 3 = 9$ Comparing the equations, one can recognise that the result must always be three times the initial number. Since three times an odd number is always odd, the result is an odd number (see Biehler and Kempen 2013, p. 89).	$130° + 50° = 180°; \frac{130°}{2} + \frac{50°}{2} = \frac{1}{2} \cdot (130° + 50°) = 90°$ The sum of the adjacent angles α and β is 180°. The bisectors split α and β into two equal angles. Accordingly, the sum of the half-angle of α and the half-angle of β is always 90°.
narrative proof	The double of an odd number is always even. Since the sum of an odd and an even number is always odd, the statement is proven.	The sum of the adjacent angles α and β is 180°. The bisectors split α and β into two equal angles. Accordingly, their sum equals half

		of 180°. Therefore, the sum of the half-angle of α and the half-angle of β is always 90°.
symbolic proof	Let a be an odd number. Then $a + 2a = 3a$. Since three times an odd number is always odd, the statement is proven (see Biehler and Kempen 2013, p. 90).	Let α and β be adjacent angles. Then $\alpha + \beta = 180°$. Accordingly, we have: $\frac{\alpha}{2} + \frac{\beta}{2} = \frac{\alpha+\beta}{2} = \frac{180°}{2} = 90°$ (see Recio and Godino 2001, pp. 85).

Table 1: Possible solutions of the proving tasks in accordance with the categorisation of Reid and Knipping (2010).

The two tasks comprise mathematical content from middle school and are meant to be easy to understand. A diagram was added to the second task, where the angles have been drawn and named for clarifying the given claim. Both tasks allow for different approaches, which might give a hint to students' personal meaning of proof.

Data collection and data analysis

This pilot study was conducted with four students in their last year in a high school (two females, average age was 20; two students attending an advanced course in mathematics and two students attending a basic course). Students' proof construction and the following interview phase were filmed in order to be able to base the analysis of the proving process not only on the participants' description of their approach but also on observations of the filmed process.

The analysis of the concrete proofs created by the participants focusses on the type of proof corresponding to the participants' approaches (see Table 1) and on the mathematical correctness. We consider a proof being correct when the arguments given are mathematically correct and link the given data with the formulated claim in a deductive manner. For describing participants' personal meaning of the notion of mathematical proof, we made use of the categories proposed by Recio and Godino (2001) (see above). However, we did not want to assign the personal meanings of the notion of proof only based on students' proof productions. We also carried out a qualitative analysis of students' responses from the interview to increase the validity of our research.

For answering the first research question, we will categorize students' proof productions and rate their correctness. (We consider a proof being correct, if the claim is logically derived from the premises. We call a proof incorrect, if at least one of the arguments used is not true (in general) or if part of the whole chain of argument is missing.) We will combine the results of students' proof productions, their proving process and their answers from the interview to describe their personal meaning on the notion of mathematical proof (research question 2).

RESULTS

In this section, we will describe and discuss the results of each participant separately to work out a uniform description for each participant. Due to the size of the paper, we had to look for a selection from students' responses to analyse their performances.

Students' proof productions and personal meanings

Participant 1 creates a generic proof for the first task and a narrative proof for the second (see Figure 1). However, the argument explicated in the generic proof is incomplete, because the fact that one addend is odd does not explain the parity of the result. In the second task, the student's use of the word parallel is not correct. Moreover, she seems to give some kind of intuitive argument, why the angle stays the same.

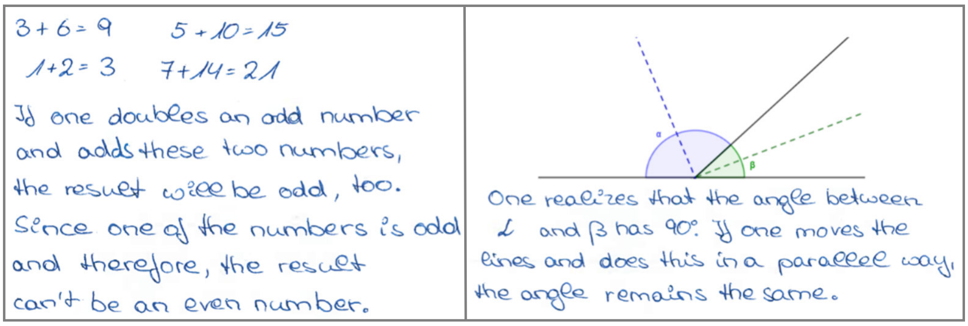

Fig. 1: Participant 1's proof productions (left: task 1; right: task 2; our translation)

When working on the first task, Participant 1 checks several examples and tries to find and to assemble arguments. She recalls her proving process as follows:

> Participant 1: So, I started with 3, added 6, because 6 is the double, and then you have 9. And then I was thinking, what will happen with other numbers, and there will always be an odd sum because when you have an odd number, when you take its double, which is an even number, and when you add an odd and an even number, you obtain an odd number. (Transcript 1; our translation)

This student uses concrete examples to find a pattern that might constitute a generic argument. Accordingly, she constructs a generic proof (see Figure 1, left). Unlike her explanation given in the interview, the explication of the argument in the written proof is incomplete. Concerning task 2, the student uses her set square to measure the angle in the graphic on the exercise sheet:

> Participant 1: I took the set square to check if the angle is 90°. […] Then I was thinking about, how one might prove this, because, one sees that the angle has 90°. I did not come up with a calculation or an equation […]. And then I was thinking: When you move this [the angle bisectors of α and β], they stay parallel; you always move them parallel; accordingly, this angle stays the same. (Transcript 1, our translation)

This approach of measuring the angle seems to be interesting, because the fact about the right angle has been mentioned explicitly in the task. Afterwards, as she recalls, she was looking for some kind of calculation. Not coming up with an equation, she tries to describe some kind of "dynamic" argument about moving the bisectors. Her 'narrative proof' is based on a visual impression and not convincing.

Participant 1's meaning of proof is beyond purely empirical evidence. She is trying to find an argument to verify the given claims in general. However, she expressed her satisfaction with her solution, because she "proved the claim given in task one". To sum up, participant 1's personal meaning of proof seems to be in line with the informal deductive-proof scheme. While looking for deductive reasoning in order to verify the given claim in general, she makes use of rather informal arguments.

Participant 2 has serious problems to understand the first proving task. Accordingly, we will not discuss his solution for task one here. In the second task, the student is finally making use of algebraic variables and equations to verify the given claim in general and he constructs a symbolic proof.

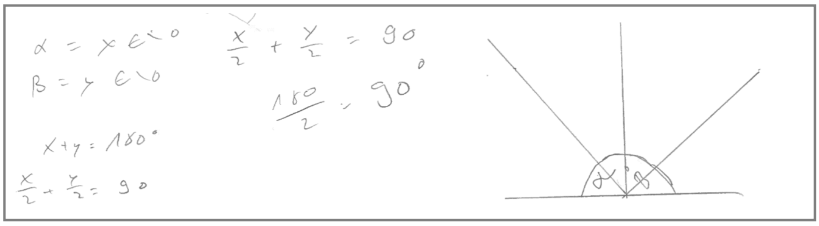

Fig. 2: Participant 2's solution to task 2

Like Participant 1, also this student uses his set square to measure the angle in the graphic on the exercise sheet. Afterwards, he creates a special case for the given claim. He explains his use of this example (the special case shown in Figure 2) as follows:

> Participant 2: OK, that's true. But why? Stop, I'll take another example. […] When I do it like this; I take 90° and both halves have 45°, and that's again 90°. A Coincidence? I didn't know. Good to know! That's really true.

Afterwards, he transmits the idea of the bisectors α and β to variables x and y:

> Participant 2: OK, α has a certain angle, called x. And for β we take y. I would say: x divided by two plus y divided by two equals 90. […] I know why it's true. Since half of 180 is 90, true? Yes, here we have 180° and half of it is 90°

Participant 2 is engaged to give a general argument and an explanation of why the claim holds in every case. He uses a special case to find a general argument. Since he is particularly making use of algebraic variables, we assign his personal meaning of proof with the formal-deductive proof scheme.

Participant 3's solution for the first task can be considered as a sketch for a generic proof; his answer to the second question seems to be a description of the given facts, not containing any argument (see Figure 3.)

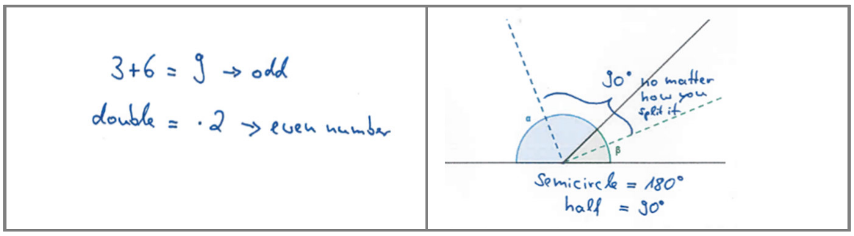

Fig. 3: Participant 3's proof productions (left: task 1; right: task 2; our translation)

In the interview, Participant 3 explains her use of the concrete example in the first task:

Participant 3: You have … to prove something … you need examples. You have to prove it and that's the evidence.[1] […] Yes, I succeeded, the presentation … to prove it. But to prove it logically, that was harder for me.

Interviewer: OK. Accordingly, how to prove it logically?

Participant 3: Yes […]. Not just based on one example. But to say that it is always like this. I guess, there'll be a logical explanation, but I just don't know it.

Here, the example is used to show that the statement is true in this special case. Interestingly, the student calls this a proof. But she also displays a second notion of proof when talking about „to prove it logically". In the second case, a proof is „not based on one example" but concerned with generality.

In the second task, this student determines the 90° angle (see Figure 3). However, she expresses dissatisfaction with her solution and her preference for mathematical symbols in the context of proving:

Interviewer: What did you not do well?

Participant 3: Maybe, I did not explain the connection, so... proving. […] I guess, there will be something like a rule or something like this. […]

Interviewer: You just mentioned that a general rule was important for you. Of what importance is the use of mathematical symbols and variables for proving?

Participant 3: It is important. It is easier to understand, the proof. So, I think, it is always important.

Participant 3 seems to be aware of the general character of mathematical proofs. However, there seems to exist two different kinds of notion of mathematical proof for her. One kind of "proof" is about empirical evidence about the truth of a statement in in some concrete cases. This view is in line with the meaning called "explanatory argumentative scheme" (see above). On the other hand, there is the "logical proof", as she calls it. This kind of proof is concerned with generality. To perform the corresponding kind of proving, the student is trying to use valid arguments and rules or formulas. In this case, the meaning of proof seems to be in line with a deductive proof scheme.

[1] In original language (german): „Man muss das ja belegen, das ist sozusagen der Beleg."

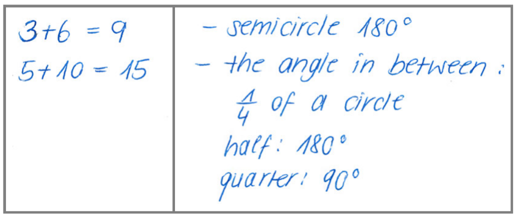

Fig. 4: Participant 4's proof productions (left: task 1; right: task 2; our translation)

Participant 4 checks two concrete examples when working on the first task. Like Participant 3, his solution of the second task is a description of the given facts without any argument (see Figure 4). He comments his approach in the first task as follows:

Participant 4: Here [in the claim], it says "always". That's what bothers me with my solution because I do not cover this, because I have only a limited number of examples. I have two, but there are infinite. In this manner, I cannot prove [it for] all. I would need a paper, where I could write down all of them. […]

Interviewer: To sum up: Would you consider your solution being a proof?

Participant 4: Yes, because I showed it with two examples. That it is…yes… as long as there is no counterexample, then I would say: "It is like this".

Concerning the task 2, the student mentions some dissatisfaction with his solution:

Participant 4: I think, it is weaker than the other […], because here again, it is written "always". And I have shown that the angle in the diagram has 90°. And when you …, yes, I did not do so well, that the angle is always about 90° […]

Participant 4 is aware of the limitation of his approach for task 1. Like Participant 3, this student seems to distinguish two different meanings of "proof". The first is concerned with illustrating the truth of a given claim with some concrete examples. The latter is in line the generality of mathematical statements. In this case, the testing of a finite number of examples cannot form a mathematical proof.

To sum up, all participants seemed to be aware of some kind of generality when dealing with the given mathematical claims. In this sense, a deductive proof scheme could be assigned to all students, when mentioning that the mere use of concrete examples without further argument cannot prove a given claim in general. However, two students also held a second view on proof simultaneously, being somehow in line with the explanatory argumentative scheme or the empirical-inductive proof scheme.

FINAL REMARKS

As one output from this pilot study, we want to highlight that the combination of proof productions and the following interview phase to reflect on one's solutions seems to be valuable to investigate students' personal meaning of mathematical proof. While some students only tested some concrete examples when working on the task, they clearly mentioned the limitation of their approach in the interview. Interestingly, two participants in this study displayed several different personal meanings of the notion of mathematical proof. Therefore, a one-to-one assignment of student and personal meaning does not seem to be possible nor desirable, because it would lead to an oversimplification of the issue. This result might give a hint, why some students call an empirical argument a “proof”. These students might use the word “proof” in a special way, being nevertheless aware of the limitation of an empirical-inductive approach. This first result has to be checked in the upcoming main study.

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