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The use of integrals for accumulation and mean values in basic electrical engineering courses

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Introduction

This paper investigates the use of integrals in a signal analysis task assigned to students in an examination in a first-year course on electrical engineering (EE). In this task, the students are to calculate several mean values of voltages necessary for describing the behavior of electrical networks using a voltmeter. We aim at understanding what kind of mathematics is required for the solution and how it is used. We used this task in a larger empirical study of several tasks. We interviewed experts in EE with regard to their understanding of the competencies needed to solve the task, studied pairs of students' solution processes, and analyzed written student solutions taken from the course exam. We focus on the interview conducted with the EE-expert, who was asked to work on the problems using the expected knowledge of good students after their second semester. The interview was done using the PARI-methodology (Hall et al., 1995): after work on the task, the expert was asked for reasons for their decisions and actions and explicitly about the required competencies.

The task highlights various practices and disparities between mathematics and engineering courses typical for such exercises. One contributing factor to these disparities is that engineering students often learn mathematics separately from the engineering courses, leading to several challenges. For instance, mathematics courses (MfE) follow a deductive conceptual structure, whereas EE-courses have an order of topics according to electromagnetic theories and often also according to traditions. Moreover, mathematical practices in EE can be characterized (Alpers, 2017) that differ from corresponding practices in mathematics courses, e. g., infinitely small lengths (dl), areas (dA) or volumes (dV) are treated as infinitesimally small quantities, which are cumulated by integration.

Theoretical background and methodology

To facilitate the study of EE-problems we have devised a model for a normative solution, the student-expert-solution (SES), which is a central tool for the analysis of solution processes and products based on the rather short solution for the correctors of the exam. The final step of its generation is an expert interview to find out the competencies and skills expected from students when solving the exercises, for more details, see Kortemeyer and Biehler (2022). With regard to the integral concept, we distinguish between different interpretations or mental models: (oriented) area, accumulation, antiderivative, as well as mean value (average), as discussed in Greefrath et al. (2021) and Thompson & Dreyfus (2017), which are relevant at school level. For our analysis, we address the following three research questions: What skills and conceptual understanding related to integrals are required for students to solve EE-exercises? What is the role of different mental models or interpretations of integrals in these models, such as oriented measure of area, accumulation, mean value? How does this necessary knowledge relate to the practices in MfE and EE?

The use of integrals in the EE course and students' prior knowledge

Considering prior knowledge, students should be rather familiar with both the antiderivative and the oriented area conceptions of the integral. The lecture notes of the EE-course provide formulas that usually are developed by using differentials and the accumulation approach. However, it is not explicitly stated whether the derivation of the formulas was covered in the EE-lecture. The lecture provides formulas for calculating means of a varying quantity using integrals, without deriving them, probably assuming such a use of integrals was covered in MfE. However, this assumption is often not accurate. It will become evident that the exercises focus on the application of the provided formulas rather than on developing them from basic laws using the accumulation aspect of integrals.

The exercise on signal analysis

The voltage $u_L(t)$ shown in the figure is applied to a coil $L = 100$mH for a duration of $T = 10$s:

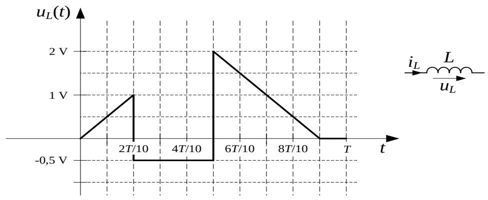

Figure 1: Given sketch of the voltage values

Based on voltage values $u_L(t)$ depicted in the diagram, the students are to calculate (1) the average value, (2) the RMS-value and (3) the rectified value of the voltage within the interval 0s to 10s. Student need to recall three relevant formulas from the lecture, and then the integral mathematically.

(1) $\bar{u} = \frac{1}{T}\int_0^T u_L(t)dt$, (2) $u_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T u_L^2(t)dt}$ and (3) $|\bar{u}| = \frac{1}{T}\int_0^T |u_L(t)|dt$

Normative solution and expert interviews

In this section, we analyze the three parts of the exercise in parallel, as they share many similarities. The function $u_L(t)$ is defined piecewise. From the MfE-course view, it would be necessary to determine (interval-wise) formulas for the function sketched in Figure 1 and then calculate the integrals in the four intervals and use the additivity theorem of the integral to calculate the integral for the whole interval from 0 to T. Students have to work with quantities and their units, which differs from those in MfE. Also, the formulas for the function in the four intervals must be specified with units, functions have to be regarded as functions between magnitudes, rather than sets of numbers as in MfE. For example, in the interval from $5T/10$ to $9T/10$, the correct formula is $u_L(t) = 4{,}5\text{V} - 1\text{V/s} \cdot t$. t represents time in seconds (s), resulting in $u_L(t)$ being measured in volt (V). Students are expected to use their school knowledge to specify the formula for the different linear or constant functions in other intervals. However, in school, setting up such formulas with units was not practiced. From these formulas, just the square or absolute values must be taken in (2) resp. (3). These leads to elementary functions, where the integral can be evaluated from the antiderivative point of view.

However, the expert proposes a different approach: He uses the area interpretation not just as an interpretation (as often in schools) but as a calculation method. The oriented area under the curve in

Figure 1 can be calculated by elementary geometric area formulas. It is moreover allowed to cut and rearrange parts while the size of the area remains invariant. To calculate the average, the expert employs a technique he calls “block shoving”. After writing down formula (1) he says: "That means we now want to calculate the two areas under here." He marks both triangles, separates the apex ($u_L(t) > 0.5$V) from the front triangle, and moves it so that a rectangle with height 0.5V is obtained, which can be calculated as 0.5V·2s = 1Vs. He continues with regard to the interval from "0.5V times 3s is -1.5Vs. And here [referring to the interval between $5T/10$ and $9T/10$] we can shove blocks again. 1V times 4s are 4Vs. A total of 3.5 seconds cancels out [he means the division by T in formula (1)], that's 0.35V.” The area has the dimension V·s that is not only the dimension of magnetic flux but can be interpreted as the magnetic flux between 0 and T. The average $\overline{u}$ is the fictious constant voltage that is needed to achieve the same magnetic flux as the signal shown in Figure 1. In geometric terms: The rectangle from 0 to T with the height of $\overline{u}$ has the same area as the oriented area under the signal function in Figure 1. This interpretation is not made explicit but underlies implicitly the calculation.

This strategy employs a combination of two conceptions of an integral: firstly, the integral itself provides the oriented area between the t-axis and the graph of the function, and secondly, the formula applied in (1) yields the mean of the cumulative values of $u_L(t)$ in the interval of integration. The cumulative conception of the integral is needed for the derivation of the formula, but it is not needed for the calculation. Instead, the expert replaces the integral by a sum of areas of geometric objects which can be calculated by known formulas without applying calculus.

Regarding the calculation of (2), the expert acknowledges its complexity, stating, “Things are a bit more complicated because you have a square and a root in there. The square makes life a bit difficult. If you have a constant value, you square it and you can still calculate it with the rectangles. With triangles it is different, because they become parabolas by squaring. It is not possible to calculate the area in one go. You really have to calculate the integral. You have to do this piece by piece, but you can move the blocks back and forth as you like, which is what I did here. That's why the lower limits are all zero. This eliminates terms, which makes it easier.” The expert acknowledges that integration cannot be avoided in this part, but he attempts to simplify this integration as much as possible by the “moving block” technique again. He shifts the shapes of the rectangle in the interval from $2T/10$ to $5T/10$ and that of the triangle in the interval $5T/10$ to $9T/10$ along the t-axis so that the lower limit of all integrations becomes zero, which affects both the upper limit of integration and the formula of the functions in the integrand. This simplifies the evaluation of the integrals, as zero can be inserted for the lower limit in each case, and the formulas of the linear functions are easier to determine.

In (3) - using the block-conception again – the expert says, that all blocks have now a positive sign because of the absolute value – but the areas that were calculated in (1) can be reused, giving 0.65V instead of 0.35V as the result. The expert justifies that by stating that the value has to be greater than in task (1), as all the blocks have to be taken positively. This can be even considered as a generic geometric proof that the integral of the absolute value of a function is always equal or bigger than the integral of the function. This is a very different kind of justification than the proof in MfE, where the monotonicity of the integral is used for justification and such geometric arguments would usually not be accepted.

Summary and recommendations

This paper presents an exercise on signal analysis that involves the use of integration to calculate different means of a process where the voltage varies. Through the analysis, we have observed several differences between the practices in MfE and EE. The calculation of the integral is done or simplified by geometric considerations and operations, based on a clear understanding of the integral as an oriented area. It is noteworthy that both the area and the average have units with a physical meaning. These findings are intriguing as one might have expected the accumulation aspect to hold greater importance in physics and EE. While this aspect is indeed relevant in the derivation of the formulas, the exercise in this examination does not assess the competence of deriving and interpreting the formula. Instead, it focuses on the application of the formulas to a given situation. Moreover, the interpretation of integration as an average is also highly relevant, despite not holding a prominent role in school calculus or in MfE courses. Considering these differences, if they persist, they could lead to improvements in both courses, fostering better coordination between MfE- and EE-aspects.

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