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"I only know the absolute value function" – About students' concept images and example spaces concerning continuity and differentiability

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We analyse students' work on a task concerning relations of various concepts of differentiability in $\mathbb{R}^n$ to find out about their concept images about continuous but non-differentiable functions other than the prototypical examples of functions with a "cusp" like the absolute value function. We identify different types of continuous, non-differentiable functions and show which types seem to be more accessible for the students than others. We use a study with sixteen students in an Analysis-II-course at a German university.

Keywords: Teaching and learning of Analysis and calculus, epistemological studies of mathematical topics.

INTRODUCTION

Differentiability and derivatives are essential topics in school and university mathematics and have been studied extensively in different contexts (e. g. Orton, 1983; Zandieh, 2000). In another article (Lankeit & Biehler, 2019), we described a task where Analysis-II-students were asked to explore the relations between different concepts of differentiability in $\mathbb{R}^n$ such as total differentiability, partial differentiability, one-sided directional differentiability and continuity. We found out that one of the implications students had the most difficulties with was the question of whether continuity implied the existence of all one-sided directional derivatives. Only one out of 31 students who handed in their written work, produced in a tutorial group meeting, stated a correct example for a function that is continuous but for which not all one-sided directional derivatives exist in $x = 0$: the function $\sqrt{|x|}$. Five of the students gave the absolute value function as an example, which is not a legitimate counterexample since all one-sided directional derivatives exist. To find out why so many students could not come up with a valid example, we will have a look at the transcripts of a subgroup of sixteen students whom we videotaped while working on this task. We will examine how they argue and what functions they consider. This analysis will provide exciting insights into these students' concept images concerning continuous and differentiable or non-differentiable functions.

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

We based our task design and analysis on Brousseau's Theory of Didactical Situations (TDS) (Brousseau, 2006). A *situation* describes the circumstances in which students find themselves concerning their *milieu* (the set of objects on hand, available knowledge and interaction with others). In this theory, we distinguish between didactical and adidactical situations. A situation is of adidactic nature if the teacher does not instruct, but students work autonomously and learn by adapting to the milieu whereas, in a didactical situation, acculturation happens through institutionalisation and devolution. For a more detailed description and a well-presented introduction of TDS, see for example, Artigue, Haspekian, and Corblin-Lenfant (2014). For the analysis of students' work on the task, we use the notions of concept image and example space. Tall and Vinner (1981, p. 152) describe the concept image as the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes". It is important to note that a concept image does not have to be coherent. It is also notable that not all parts of the concept image are evoked at the same time. An essential element of the concept image is the related example space (Goldenberg & Mason, 2008) which contains examples, non-examples and counterexamples for the concept. We consider the concept image as a part of students' milieu when working in a specific situation.

Not much is known about university students' concept image of differentiability or non-differentiable functions after they have been taught a formal approach as compared to the situation in school where arguing with interpretations like "tangent slope" is more common. Viholainen (2008) presents the case of a student who claimed that several piecewise-defined functions with jumps were differentiable because it was "constant where the jump occurred" so that the derivative in that point "was zero". However, that functions whose graphs depict "corners" could not be differentiable was clear to him. This example illustrates that this student's concept image concerning non-differentiable functions is not complete, and especially it was not clear to him that differentiability implies continuity. A problem for students correctly linking differentiability and continuity is also reported by Juter (2012) and Duru, Köklü, and Jakubowski (2010) who found that many students believed continuity implied differentiability in the one-dimensional case. Klymchuk (2005) showed (with a small sample) that in a group of students where counterexamples were not used regularly and explicitly in the lecture, less than half of the students were able to sketch a graph that was continuous, looked smooth and was at one point not differentiable.

RESEARCH QUESTIONS

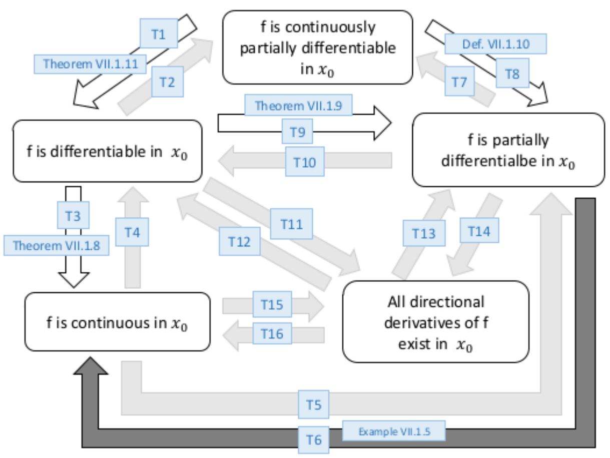
The broader aim of the whole study is to improve our understanding of students' difficulties concerning the different concepts of multivariable differentiability and their connections to one-dimensional differentiability. This understanding can inform the teaching of these topics. The understanding of multivariable differentiability cannot be separated entirely from that of one-dimensional differentiability because it builds on it.

In this article, we are interested in the following research questions: What makes the task of deciding whether continuity implies one-sided directional differentiability difficult for the students: Do difficulties occur in translating the item into the one-dimensional case or do the problems lie in insufficient concept images for non-differentiable functions in the one-dimensional case? What kinds of functions do the students consider when trying to find an example and what can we learn about students' concept images and example space concerning differentiability and continuity in the one-dimensional case from their work on this task? What can be done to improve students' performance on this task?

METHODOLOGY AND STUDY DESIGN

Our study took place in an Analysis-II-course (which is, from an international perspective, more on the level of upper-division proof-oriented Real Analysis courses in the US than typical lower-division Calculus courses). We did not influence the lecture the students participated in but designed two tasks concerning differentiability in $\mathbb{R}^n$ in cooperation with the lecturer and his teaching assistant. The students (second or higher semester, depending on their study program) worked on these tasks in two of their weekly tutorial group meetings. The task that we are concerned with here is part f) of the task shown in figure 1, for a more detailed description of the task and the design principles (guided by different "task potentials" that Gravesen, Grønbæk, and Winsløw (2016) formulated building on TDS) see Lankeit & Biehler (2019).

In this diagram it is marked which logical relations should be known from the lecture. (White means the implication is valid, dark grey means the implication is invalid.)



b) The function

$$f:\mathbb{R}^2\to\mathbb{R},\quad f(x):=\begin{cases}\frac{x_1x_2}{x_1^2+x_2^2}, & \text{if } x=(x_1,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}\\ 0, & \text{if } x=(0,0)\end{cases}$$

was used to reason that implication T6 is invalid. Why does this example mean the implication is not true?

c) Next, consider the converse directions of the by lecture valid implications, thus the implications T10, T7, T2 and T4. Tick whether they are true or false and prove your claims.

d) How do differentiability and the existence of all one-sided directional derivatives correlate? Investigate whether the implications T11 and T12 are valid.

e) Examine the logical relations between partial differentiability and the existence of all one-sided directional derivatives by checking whether T13 and T14 are true.

f) Is it possible to deduce from continuity in a point x_0 that a function is partially differentiable or all one-sided directional derivatives exist in this point?
(I.e., investigate the implications T5 and T15.)

Figure 1: The discussed task (translated by the first author).

We chose eight pairs of students to work on this task not in their usual tutorial group but separately in a situation where the first author acted as a tutor. The selected students were in their second or higher semester and studied Mathematics (4 students), Computer Science (1 student) or were pre-service teachers (11 students). Each group was filmed while working on this task. The written work they produced while working on the task was collected as well. The videos were transcribed after collecting the data. The transcripts were then analysed concerning our research questions. Transcripts shown in this article are translated from German by the first author.

The situation could be (and was in all of the cases) transformed from an adidactical to a didactical situation when the tutor stepped in, asked questions or gave hints. Since the time we had for the interviews was limited and this was the last task, we did not in all cases let the students think on their own or allow them "walk in the wrong direction" as long as we would have done for other tasks. Therefore, we can only conclude that some functions might not be as readily available in the students' example spaces as we would like them to be, and not that they are not at all contained in the example spaces.

EPISTEMOLOGICAL ANALYSIS AND A PRIORI ANALYSIS

The question of whether or not continuity implies the existence of all one-sided directional derivatives translates in the one-dimensional case to the question of whether continuity implies right- and left-sided differentiability. There are different reasons for functions not to be differentiable in the one-dimensional situation. As known, a function $f: \mathbb{R} \to \mathbb{R}$ is differentiable in $x_0 \in \mathbb{R}$ if the limit $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists. This limit does not exist if (a) the right- and left-sided limits exist but are not the same, (b) the term tends to infinity (for $h \searrow 0$, $h \nearrow 0$ or both) or (c) oscillatory behaviour (from at least one side) occurs in a way that makes the limit not exist. Case (a) happens for example in the absolute value function and means that the graph of the function has some sort of "corner" or "cusp", i.e. an abrupt change of the slope. Case (b) means that a tangent line to f at the point x_0 is vertical. This kind of behaviour can be found for example at $x_0 = 0$ in the cubic root function $f: \mathbb{R} \to \mathbb{R}, f(x) = \sqrt[3]{x}$ or in a suitably continued square root function, e. g. the functions $f: \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{|x|}$ or $f(x) = -\sqrt{-x}$ for $x \in \mathbb{R}^{<0}$ and $f(x) = \sqrt{x}$ for $x \in \mathbb{R}^{\geq 0}$. Case (c) occurs for example in the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$, at the point $x_0 = 0$. A function that is not differentiable because of (a) still has both one-sided directional derivatives. In cases (b) or (c), at least one of the one-sided directional derivatives does not exist. This means that the standard example for a continuous but not differentiable function, i. e. the absolute value function, cannot be used to contradict the implication "continuity implies one-sided directional differentiability" (as well as any other function that is not differentiable because of a cusp). However, the example functions for cases (b) and (c) provide valid counterexamples since all of them are continuous.

In our a priori analysis, we expected that students would first think of the absolute value function and then quickly come to the conclusion that this is not a counterexample because the one-sided derivatives in 0 exist (although they are not the same). We expected them to try to come up with other functions $\mathbb{R} \to \mathbb{R}$ that are continuous but not differentiable and assumed that most students would – maybe with some help – think of some sort of a root function. A more detailed a priori analysis was done in Lankeit and Biehler (2019). The example space concerning non-differentiable functions was not explicitly cared for in the lectures. As usual, the absolute value function was given as an example for a non-differentiable, continuous function. Examples from the case (b) or (c) were not addressed in particular in the analysis I course preceding the discussed Analysis II course. The function $\sqrt[n]{x}$ on $(0, \infty)$ for $n \in \mathbb{N}$ was used as an example for a differentiable function as well as the function $f(x) = x^3 \cdot \sin\left(\frac{1}{x}\right)$ for $x \in \mathbb{R} \setminus \{0\}$, $f(0) = 0$.

RESULTS

We will now give an overview over the groups' work on the question whether or not continuity implied the existence of (one-sided) directional derivatives with an emphasis on our research questions which different functions they debated and what we can learn about their concept images. For space reasons, we will not show detailed case studies for the groups independently but rather give summaries over all of the groups concerning example functions they mentioned and students' reactions to the idea of a vertical tangent.

When they started looking for an example function, all of the pairs considered functions $\mathbb{R} \to \mathbb{R}$ and had no trouble translating the question into the question whether continuity implied the existence of the limits $\lim_{h \searrow 0} \frac{f(x_0+h)-f(x_0)}{h}$ and $\lim_{h \searrow 0} \frac{f(x_0-h)-f(x_0)}{h}$. Some of them started by actually trying to find a proof for the implication but recognised errors in their "proofs" themselves or with the help of the tutor. In this article, we will concentrate on the attempts to falsify the statement. Some of the groups also tried to use the logical structure of the diagram given in the task (see figure 1) which is something we found students doing for many of the implications, see Lankeit and Biehler (2019). It is not possible to use this diagram-based strategy successfully for this task if all earlier implications have been assigned the correct truth values.

Example functions the groups used

We will now have a look at example functions the groups came up with themselves, without the tutor hinting at a specific function (e. g. by saying the name, sketching the graph or asking for a function's inverse function). We will group the examples the different pairs came up with by the different cases ((a)-(c)) we described above. We additionally add the group (d) of discontinuous functions that were wrongly discussed as counterexamples, even though discontinuous functions could also be grouped into the cases (a)-(c). Still, it seems helpful to differentiate between continuous and discontinuous examples because their non-continuity makes them unsuitable as counterexamples in this task. We also added a category of differentiable functions (e) that students wrongly mentioned as candidates for non-examples. The groups came up with the following examples [1] in the five categories (a)-(e) on their own:

(a) $|x|^*$ (7 groups), $f(x) = \begin{cases} x^2, x < 0, \\ \sin(x), x \geq 0, \end{cases}$ (1 group), $f(x) = \begin{cases} 0, x \leq 0, \\ \text{x}, x > 0, \end{cases}$ (1 group), other (2 groups)

(b) (vertical tangent): None

(c) $x \sin\left(\frac{1}{x}\right)$ (2 groups)

(d) Step function* (1 group), $\frac{xy}{x^2+y^2}$* (1 group), other (1 group)

(e) $e^{x^{**}}$ (2 groups), x^2* (2 groups), $x^2 \sin\left(\frac{1}{x}\right)$* (1 group), $\frac{1}{x}$ (1 group), $\frac{1}{x}^{**}$ (1 group), $\tan\left(\pi x - \frac{\pi}{2}\right)$** (1 group), other (1 group)

When a group immediately (i.e. in the same or one of the two following turns) after stating the example recognised that it is not a suitable counterexample, it is marked with "*". The tag "**" is used to mark examples the students came up with after the tutor introduced the idea of a vertical tangent. "Other" means the group talked about some other functions from the respective category without explicitly specifying it. It should be noted that none of the groups came up with an example of case (b) ("vertical tangent"), but two groups found an example of case (c). All groups who considered the absolute value function as a counterexample immediately realised it was not a suitable one. Only one group did not discuss this function (or any other function with a cusp).

Most of the groups at first only came up with functions with cusps or only the absolute value function. Some of them commented on this like the following quotes:

Peter: Can you come up with anything? Because mine [my example for a continuous, non-differentiable function] is the absolute value function by default. Because there it is nice that one has the visual evidence why it doesn't work.

The tutor asked five of the groups for reasons why a function could be continuous but non-differentiable. All of them only mentioned functions with cusps and in some cases, discontinuous functions, similar to group 1:

Tim: What other functions that are not differentiable do I know? […][2] It would have to be functions that have some kind of cusp, right?

Michael: Yes.

Tutor: Have a cusp, or what else would be possible?

Tim: Have a gap. But then it would not be continuous.

Group 8 reacted similarly:

Carl: Then I don't know any other class of functions that is non-differentiable and continuous, other than cusps.

Only the groups 3 and 7 came up with an example that could be successfully used to falsify the statement "continuity implies the existence of all one-sided directional derivatives". Both groups used the same example, the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$, at the point $x_0 = 0$. How they came up with it was different in those cases: The tutor had shown the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 \cdot \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$, to group 3 in part c) of the considered task as an example for a differentiable function for which not all (partial) derivatives are continuous. They remembered having seen this example before but said they wouldn't have hit on it by themselves. After having seen this in the earlier task, they had a look at it again in part f) when they were looking for a function that is continuous but does not have all one-sided directional derivatives. They noted the function was continuous and then wanted to check out whether the one-sided directional derivatives existed. When asked by the tutor what this function had been an example for, they realised it was differentiable which implied the existence of all directional derivatives. They then decided to modify the function:

Sophie: What happens if we only take x here? Because she [the tutor] said before it's not differentiable if we only take x.

In group 7, the tutor had not introduced this or a related example before. Marc mentioned $\frac{\sin(x)}{x}$ wanting to find "that function that oscillates around zero the closer we get to zero" and finally developed (guided by the tutor's questions) the above function. It is clear from what he said that he had seen this or a related function before.

In the other six groups, the tutor introduced the idea of using a variant of the square root function (see above), either by saying the name of the function, sketching the graph or hinting at it by asking for the inverse function of the quadratic function. Depending on the time, the groups then checked themselves or together with the tutor that the one-sided directional derivatives in 0 do not exist.

The idea of a "vertical tangent."

The tutor gave different hints for each group, depending on what they needed. In groups 1 and 4, the tutor introduced the idea of a function with a "vertical tangent", or that is very steep at some point. In the first group, Michael answered with the exponential function but immediately stated that this function is, in fact, differentiable. In the fourth group, Laura mentioned a function with an asymptote, a function that approaches the y-axis like $\frac{1}{x}$, and later the exponential function. In group 8, David explained that another function they had discussed in a previous task (which was not continuous) did not have all one-sided directional derivatives in the following way, thus introducing the idea of an infinite slope himself:

David: The problem was that there wasn't really a slope but rather a steep ascent tending to infinity. But that function was not continuous.

When asked what this would mean visually for the graph of a function $\mathbb{R} \to \mathbb{R}$, David and Carl talked about functions with bounded domain and unbounded codomain and mentioned a bijective tangent function (we believe they meant $f\colon(-1,1) \to (-\infty, \infty), x \mapsto \tan\left(\pi x - \frac{\pi}{2}\right)$) but realised there is not one specific point where the slope is infinite. When asked for a function with a particular point where the slope is infinite, the following dialogue happened:

Carl: Some kind of… a vertical line, somehow.

Tutor: Do you know any function that behaves like that?

David: Yes, a step function, but that is not continuous. (laughs)

Carl: Exactly. It is either not continuous or not well-defined in that sense.

The second group worked with the definition of one-sided directional derivatives. It came up with the idea that the limit should be $\pm\infty$ so that the one-sided directional derivative does not exist. The first example they tried was $\frac{1}{x}$. They did not translate this into the idea of a vertical tangent and did not advance this idea but tried other strategies next.

On the other hand, after discussing $\sqrt{|x|}$, when the tutor asked the fifth group whether they could have seen the function is not differentiable in 0 before calculating that the limit of the difference quotient does not exist, Peter answered in the following way:

> Peter: […] We don't have an unambiguous way to find the tangent is zero. […] Respectively, the tangent gets steeper and steeper and steeper […] until it would be vertical, which is impossible.

At first, he uses the argumentation he used to explain why the absolute value function is not differentiable in 0. Still, he then realises that the square root function is a different case and gives the idea of a vertical tangent himself. It can also be seen from this excerpt that he does not accept a vertical tangent as a tangent to the function graph.

DISCUSSION

Students correctly had the idea to look for counterexamples in the one-dimensional case. However, finding examples for functions $\mathbb{R} \to \mathbb{R}$ that are continuous but not differentiable from the left and right side was problematic. The students' remarks show that for a non-negligible part, the accessible example space concerning continuous but non-differentiable functions contains only the absolute value function. Most of the groups seemed to be limited to functions with a cusp when thinking about continuous, non-differentiable functions. This finding is not very surprising since the absolute value function is the prototypical function for a non-differentiable, continuous function that was shown to them in the Analysis I lecture preceding the discussed Analysis II course when differentiability was introduced. Differing from findings in the analysis of students' written solutions (Lankeit & Biehler, 2019), the problem of using the absolute value function as a counterexample did not occur. Most groups discussed this function but quickly realised it was not a suitable counterexample. Additionally, a broader range of functions was considered as possible examples. Both differences might be due to the different setting: The didactical contract is slightly different when under individual observation than in the usual group work. It can be assumed that the students had a greater need for a solution. Additionally, while the situation was purely adidactical for the students in their usual tutor group meeting since the tutors were advised not to help them, the groups in this video study had the help of the tutor who interacted with them.

None of the groups came up with an example from the group (b) by themselves, a function that has a "vertical tangent" at a point. Two groups mentioned that the limit might not exist because it is $\pm\infty$ but could not find an example. The idea of a "vertical tangent" evoked images of functions that become steeper when approaching infinity or a pole but not of functions with "infinite slope" at one point. This idea seemed to be new for the students, which shows that this type of behaviour of functions needs to be addressed more explicitly. It is, however, not unexpected since the tangent to a graph at a point where the function is not differentiable is often not defined distinctly. In Biza, Christou, and Zachariades (2008), the task where a vertical tangent to a point of a curve needed to be drawn was solved successfully by only 33 % of the students. Problems with drawing the tangent to the graph of the function $\sqrt{|x|}$ in 0 are documented by Vinner (2002) in the school context as well, showing that case (b) is not easy for learners. Peter's explanation "until it would be vertical which is impossible" additionally shows that vertical tangents are perceived as "not allowed". The spontaneous extension of the notion of tangent should not be expected. Therefore, the hint to think about "vertical tangents" proved to be not helpful to the students. When trying to bring students to think about functions from group (b), it might be more suitable to work in a symbolic rather than a graphical way and lead the students to think about why the limit of the difference quotient might not exist.

The fact that the rather "strange" oscillating sine-function was more accessible for the students than a root function was surprising to us. The students had seen this example in an additional, voluntary task at the end of their Analysis I course (not in the lecture), and not much time was spent on it. A possible explanation is the following: A problem with the square root function is that it is defined only on $[0, \infty)$. Therefore, differentiability is often, as in the Analysis I course preceding the discussed class, only examined on $(0, \infty)$, leading students to remember this as an example for a differentiable function and not thinking about the vertical tangent in 0. The variants of the square root function mentioned above were not discussed in the lecture, and neither was $\sqrt[3]{x}$. Additionally, while the oscillating sine-function is – if introduced – always framed as a strange function serving as a counterexample to something, the square root function might be considered a too "normal" function to even consider it as a counterexample. Therefore, one should probably not expect students to invent functions from group (c) themselves without help but, if they have already seen a variant of $\sin\left(\frac{1}{x}\right)$, it might be easier to find these example functions than suitable modifications of the square root function.

These findings suggest that the students' example space as part of their milieu when in the situation of solving this task is not rich enough. These difficulties can be met in different ways. One way is addressing the cases (b) and (c) and not only the absolute value function when discussing continuous, non-differentiable functions in Analysis I. Another is enriching the milieu for this situation by hints and preceding tasks helping the students explore different cases why functions might not be differentiable and find example functions. It might be more suitable to lead them towards the group (c) than the group (b). Hinting at specific functions enables the students to solve the task by calculating that the one-sided derivatives do not exist, but this does not improve their understanding of their concept image and should therefore not be preferred.

NOTES

1. The functions are shortened for space reasons, of course, "xsin $\left(\frac{1}{x}\right)$" actually means the function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$ etc.

2. If transcripts contain "[…]", this means that we omitted a (not relevant) part from the discussion in the transcript here to shorten the paragraph. In contrast, "(…)" means there was a pause.

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